Mods Calculus

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0.0 Introduction

These notes were written in 2007 to accompany the Mods lecture courses of *Calculus of One Variable* and *Calculus of Two or More Variables*. These are mainly introductory courses in ordinary and partial differential equations but they also include other aspects of multivariate calculus. The notes occasionally digress beyond the syllabus, but such sections or comments will be clearly marked with an asterisk. They also include a wide range of exercises which can be found at the end of each section. The syllabuses for the two courses (as of Michaelmas 2007) are below. (The bullet points correspond to how the material has been arranged in chapters within these notes.)

Syllabus for Calculus of One Variable (6 lectures)

- Standard integrals, integration by parts.
- Definition of order of an ODE – example of separation of variables.
- General linear homogeneous ODEs: integrating factor for first order linear ODEs, second solution when one solution known for second order linear ODEs.
- First and second order linear ODEs with constant coefficients. General solution of linear inhomogeneous ODE as particular solution plus solution of homogeneous equation. Simple examples of finding particular integrals by guesswork. Systems of linear coupled first order ODEs. The calculation of determinants, eigenvalues and eigenvectors.

Syllabus for Calculus of Two or More Variables (10 lectures)

- Introduction to partial derivatives.
- Chain rule, change of variable; examples to include plane polar coordinates. Examples of solving some simple partial differential equations (e.g. $f_{xy} = 0$, $yf_x = xf_y$).
- Jacobians for two variable systems, calculations of areas including basic examples of double integrals.
- Gradient vector; normal to surface, directional derivative.
- Critical points and classification using directional derivatives (non-degenerate case only).
- Laplace's and Poisson's equation, including change of variable to plane polar coordinates and circularly symmetric solutions. The wave equation in two variables, including derivation of general solution. [This chapter appears earlier in the notes.]

Other Reading:

- D. W. Jordan & P. Smith, *Mathematical Techniques*, 3rd Edition, Oxford (2002)
- Erwin Kreyszig, *Advanced Engineering Mathematics*, 9th Edition, Wiley (2005).

If you have any suggestions on how to improve these notes, or spot any errors, feel free to email the author at *earl@maths.ox.ac.uk*

1. STANDARD INTEGRATION TECHNIQUES

1.1 Trigonometric Substitutions

We begin by looking at how to approach integrals such as

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{4-2x-x^2}} \quad \text{and} \quad \int_0^1 \frac{x\,\mathrm{d}x}{x^2+2x+2}.$$

You may recall similar but simpler integrals from A-level such as

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} \quad \text{and} \quad \int \frac{\mathrm{d}x}{1+x^2}.$$

To determine these we used trigonometric substitutions based on the identities

$$\sin^2\theta + \cos^2\theta = 1, \qquad \tan^2\theta + 1 = \sec^2\theta, \qquad 1 + \cot^2\theta = \csc^2\theta.$$

- **The principle being in each case that we would substitute for x in $1-x^2$ or in $1+x^2$ the correct trigonometric function to leave us with a single square**.

So in the first integral we could use $x = \sin\theta$ as $1 - \sin^2\theta = \cos^2\theta$ and obtain

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \int \frac{\mathrm{d}(\sin\theta)}{\sqrt{1-\sin^2\theta}} = \int \frac{\cos\theta\ \mathrm{d}\theta}{\cos\theta} = \int \mathrm{d}\theta = \theta + \text{const.} = \sin^{-1}x + \text{const.}$$

Note that $x = \cos\theta$ would have worked just as well.

Some similar standard trigonometric integrals are

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \sin^{-1}x + \text{const.}$$
$$\int \frac{\mathrm{d}x}{1+x^2} = \tan^{-1}x + \text{const.}$$
$$\int \frac{\mathrm{d}x}{\sqrt{x^2+1}} = \ln\left(x + \sqrt{x^2+1}\right) + \text{const.} = \sinh^{-1}x + \text{const.}$$
$$\int \frac{\mathrm{d}x}{\sqrt{x^2-1}} = \ln\left|x + \sqrt{x^2-1}\right| + \text{const.} = \cosh^{-1}x + \text{const.}$$
$$\int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} = -\sqrt{1-x^2} + \text{const.}$$
$$\int \frac{x\,\mathrm{d}x}{\sqrt{x^2\pm1}} = \sqrt{x^2\pm1} + \text{const.}$$
$$\int \frac{x\,\mathrm{d}x}{1+x^2} = \frac{1}{2}\ln\left(1+x^2\right) + \text{const.}$$

If the fifth, sixth and seventh integrals are not apparent by inspection then a substitution $u = x^2$ would help. For those with knowledge of the hyperbolic functions the third and fourth integrals can be done more easily using the identity

$$\cosh^2 t = 1 + \sinh^2 t.$$

For more on hyperbolic functions see Section 1.3 at the end of this chapter.

By completing the square (and making a substitution if desired) every integral of the form

$$\int \frac{Ax+B}{Cx^2+Dx+E}\,\mathrm{d}x \quad \text{and} \quad \int \frac{Ax+B}{\sqrt{Cx^2+Dx+E}}\,\mathrm{d}x$$

can be broken down into one of the previous forms. We return now to the integrals given at the start of the chapter.

Example 1 *Determine*

$$I_1 = \int_0^1 \frac{\mathrm{d}x}{\sqrt{4-2x-x^2}} \quad \textit{and} \quad I_2 = \int_0^1 \frac{x\,\mathrm{d}x}{x^2+2x+2}.$$

Solution.

$$\begin{aligned}
I_1 &= \int_0^1 \frac{\mathrm{d}x}{\sqrt{4-2x-x^2}} \\
&= \int_0^1 \frac{\mathrm{d}x}{\sqrt{5-(1+x)^2}} \qquad \text{[completing the square]} \\
&= \int_1^2 \frac{\mathrm{d}u}{\sqrt{5-u^2}} \qquad \text{[substituting } u=1+x\text{]} \\
&= \int_{1/\sqrt{5}}^{2/\sqrt{5}} \frac{\sqrt{5}\mathrm{d}v}{\sqrt{5-5v^2}} \qquad \left[\text{substituting } u=\sqrt{5}v\right] \\
&= \left[\sin^{-1} v\right]_{1/\sqrt{5}}^{2/\sqrt{5}} \\
&= \sin^{-1}\frac{2}{\sqrt{5}} - \sin^{-1}\frac{1}{\sqrt{5}} \approx 0.6435
\end{aligned}$$

The second integral I_2 is left as Exercise 8. ■

1.2 Integration by Parts

Integration by parts (IBP) can be used to tackle products of functions, but not just any product. Suppose we have an integral

$$\int f(x)\,g(x)\ \mathrm{d}x$$

in mind. In the main:

- **This will be approachable with IBP if one of these functions integrates/differentiates, perhaps repeatedly, to something simpler, whilst the other function differentiates/integrates to something of the same kind.**

Typically then $f(x)$ might be a polynomial which, after differentiating enough times, will become a constant; $g(x)$ on the other hand could be something like e^x, $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, all of which are functions which continually integrate to something similar. This remark reflects the nature of the formula for IBP which is:

Theorem 2 ***(Integration by Parts)*** *Let F and G be functions with derivatives f and g. Then*

$$\int F(x)\,g(x)\ \mathrm{d}x = F(x)\,G(x) - \int f(x)\,G(x)\ \mathrm{d}x.$$

IBP takes the integral of a product and leaves us with another integral of a product — but as we commented above, the point is that $f(x)$ should be a simpler function than $F(x)$ was whilst $G(x)$ should be no worse a function than $g(x)$ was.

Proof. The proof is simple — we just integrate the product rule of differentiation below, and rearrange

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F(x)\,G(x)\right) = F(x)\,g(x) + f(x)\,G(x)\,.$$

■

Example 3 *Determine*

$$\int x^2 \sin x \ \mathrm{d}x \quad \textit{and} \quad \int_0^1 x^3 e^{2x} \ \mathrm{d}x.$$

Solution. Clearly x^2 will be the function that we need to differentiate down, and $\sin x$ is the function that will integrate "in-house". So we have, with *two* applications of IBP:

$$\begin{aligned}
\int x^2 \sin x \ \mathrm{d}x &= x^2(-\cos x) - \int 2x(-\cos x) \ \mathrm{d}x \quad \text{[IBP]} \\
&= -x^2\cos x + \int 2x\cos x \ \mathrm{d}x \quad \text{[rearranging]} \\
&= -x^2\cos x + 2x\sin x - \int 2\sin x \ \mathrm{d}x \quad \text{[IBP]} \\
&= -x^2\cos x + 2x\sin x - 2(-\cos x) + \text{const.} \\
&= \left(2 - x^2\right)\cos x + 2x\sin x + \text{const.} \quad \text{[rearranging]}
\end{aligned}$$

In a similar fashion

$$\begin{aligned}
\int_0^1 x^3 e^{2x} \ \mathrm{d}x &= \left[x^3\frac{e^{2x}}{2}\right]_0^1 - \int_0^1 3x^2\frac{e^{2x}}{2} \ \mathrm{d}x \quad \text{[IBP]} \\
&= \frac{e^2}{2} - \left(\left[3x^2\frac{e^{2x}}{4}\right]_0^1 - \int_0^1 6x\frac{e^{2x}}{4} \ \mathrm{d}x\right) \quad \text{[IBP]} \\
&= \frac{e^2}{2} - \frac{3e^2}{4} + \left[6x\frac{e^{2x}}{8}\right]_0^1 - \int_0^1 6\frac{e^{2x}}{8} \ \mathrm{d}x \quad \text{[IBP]} \\
&= \frac{-e^2}{4} + \frac{3e^2}{4} - \left[\frac{6e^{2x}}{16}\right]_0^1 = \frac{e^2}{8} + \frac{3}{8}.
\end{aligned}$$

■

This is by far the main use of IBP, the idea of eventually differentiating out one of the two functions. There are other important uses of IBP which don't quite fit into this type. These next two examples fall into the original class, but are a little unusual: in these cases we choose to integrate the polynomial factor instead as it is easier to differentiate the other factor. This is the case when we have a logarithm or an inverse trigonometric function as the second factor.

Example 4 *Evaluate*

$$\int (2x-1)\ln\left(x^2+1\right) \ \mathrm{d}x \quad \textit{and} \quad \int \left(x^2-4\right)\tan^{-1} x \ \mathrm{d}x.$$

Solution. In both cases the second factor looks rather daunting, certainly to integrate, but each factor differentiates nicely; recall that

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x} \quad \text{and that} \quad \frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \frac{1}{1+x^2}.$$

So if we apply IBP to the above examples then we get

$$\int (2x-1)\ln\left(x^2+1\right) \ \mathrm{d}x = \left(x^2-x\right)\ln\left(x^2+1\right) - \int \left(x^2-x\right)\frac{2x}{x^2+1} \ \mathrm{d}x,$$

and

$$\int \left(3x^2-4\right)\tan^{-1}x\ \mathrm{d}x = \left(x^3-4x\right)\tan^{-1}x - \int \left(x^3-4x\right)\frac{1}{x^2+1}\ \mathrm{d}x.$$

To calculate the integrals

$$\int \frac{2x^3-2x^2}{x^2+1}\ \mathrm{d}x \quad \text{and} \quad \int \frac{x^3-4x}{x^2+1}\ \mathrm{d}x$$

we need to divide the denominator into the numerator and then use the previous list of standard integrals. The second integral is left to Exercise 9. To determine the first integral, we need to divide x^2+1 into $2x^3-2x^2$, either by inspection or by long division. Note

$$\begin{array}{r|rrrr} & & & 2x & -2 \\ \hline x^2+1 & 2x^3 & -2x^2 & & \\ & 2x^3 & & +2x & \\ \hline & & -2x^2 & -2x & \\ & & -2x^2 & & -2 \\ \hline & & & -2x & +2 \end{array}$$

so that

$$2x^3-2x^2 = (2x-2)\left(x^2+1\right)+(-2x+2).$$

Hence

$$\begin{aligned} \int \frac{2x^3-2x^2}{x^2+1}\ \mathrm{d}x &= \int \left(2x-2+\frac{-2x}{x^2+1}+\frac{2}{x^2+1}\right)\ \mathrm{d}x \\ &= x^2-2x-\ln\left(x^2+1\right)+2\tan^{-1}x+\text{const.} \end{aligned}$$

■

In the same vein as this we can use IBP to integrate functions which, at first glance, don't seem to be products — this is done by treating a function $F(x)$ as the product $F(x)\times 1$.

Example 5 *Evaluate*

$$\int \ln x\ \mathrm{d}x \ \textit{and} \int \tan^{-1}x\ \mathrm{d}x.$$

Solution. With IBP we see (integrating the 1 and differentiating the $\ln x$)

$$\int \ln x\ \mathrm{d}x = \int 1\times\ln x\ \mathrm{d}x = x\ln x - \int x\frac{1}{x}\ \mathrm{d}x = x\ln x - \int \mathrm{d}x = x\ln x - x + \text{const.}$$

and similarly

$$\int \tan^{-1}x\ \mathrm{d}x = \int 1\times\tan^{-1}x\ \mathrm{d}x = x\tan^{-1}x - \int x\frac{1}{1+x^2}\ \mathrm{d}x = x\tan^{-1}x - \frac{1}{2}\ln\left(1+x^2\right)+\text{const.}$$

■

Sometimes both functions remain "in-house", but we eventually return to our original integrand.

Example 6 *Determine*

$$\int e^x\sin x\ \mathrm{d}x.$$

Solution. Both of these functions now remain in-house, but if we apply IBP twice, integrating the e^x and differentiating the $\sin x$, then we see

$$\begin{aligned} \int e^x\sin x\ \mathrm{d}x &= e^x\sin x - \int e^x\cos x\ \mathrm{d}x \\ &= e^x\sin x - \left(e^x\cos x - \int e^x(-\sin x)\ \mathrm{d}x\right) \\ &= e^x(\sin x-\cos x) - \int e^x\sin x\ \mathrm{d}x. \end{aligned}$$

We see that we have returned to our original integral, and so we can rearrange this equation to get

$$\int e^x \sin x \, \mathrm{d}x = \frac{1}{2} e^x (\sin x - \cos x) + \text{const}.$$

■

When IBP needs to be applied repetitively to determine an integral it can often make sense to consider the general case and set up a **reduction formula**. For example, in order to calculate

$$\int \cos^7 \theta \, d\theta$$

we first set

$$I_n = \int \cos^n \theta \, \mathrm{d}\theta,$$

and we will aim to write I_n in terms of other I_k where $k < n$, eventually reducing the problem to calculating I_0, or I_1 say, which are simple integrals. Using IBP we see

$$\begin{aligned} I_n &= \int \cos^{n-1} \theta \times \cos\theta \, \mathrm{d}\theta \\ &= \cos^{n-1}\theta \sin\theta - \int (n-1) \cos^{n-2}\theta \, (-\sin\theta) \sin\theta \, \mathrm{d}\theta \\ &= \cos^{n-1}\theta \sin\theta + (n-1) \int \cos^{n-2}\theta \left(1 - \cos^2\theta\right) \, \mathrm{d}\theta \\ I_n &= \cos^{n-1}\theta \sin\theta + (n-1)\left(I_{n-2} - I_n\right). \end{aligned}$$

Employing the identity $\sin^2\theta = 1 - \cos^2\theta$ returns the integral to ones involving powers of $\cos\theta$ only. Rearranging the last equation to make I_n the subject we see

$$I_n = \frac{\cos^{n-1}\theta \sin\theta}{n} + \frac{n-1}{n} I_{n-2}.$$

With this reduction formula I_n can be rewritten in terms of simpler and simpler integrals until we are left only needing to calculate I_0, if n is even, or I_1, if n is odd — both these integrals are easy to calculate.

Example 7 *(See also Exercise 13.) Calculate*

$$I_7 = \int \cos^7 \theta \, \mathrm{d}\theta.$$

Solution. Repeatedly using the reduction formula above, we see

$$\begin{aligned} I_7 &= \frac{\cos^6\theta \sin\theta}{7} + \frac{6}{7} I_5 \\ &= \frac{\cos^6\theta \sin\theta}{7} + \frac{6}{7}\left(\frac{\cos^4\theta \sin\theta}{5} + \frac{4}{5} I_3\right) \\ &= \frac{\cos^6\theta \sin\theta}{7} + \frac{6\cos^4\theta \sin\theta}{35} + \frac{24}{35}\left(\frac{\cos^2\theta \sin\theta}{3} + \frac{2}{3} I_1\right) \\ &= \frac{\cos^6\theta \sin\theta}{7} + \frac{6\cos^4\theta \sin\theta}{35} + \frac{24\cos^2\theta \sin\theta}{105} + \frac{48}{105} \sin\theta + \text{const}. \end{aligned}$$

■

Example 8 *Calculate*

$$\int_0^1 x^3 e^{2x} \, \mathrm{d}x$$

Solution. We already met this integral in Example 3. We can approach this in a simpler yet more general fashion by setting up a reduction formula. For a natural number n, let

$$J_n = \int_0^1 x^n e^{2x} \, \mathrm{d}x.$$

We can use integration by parts to show

$$J_n = \left[x^n \frac{e^{2x}}{2} \right]_0^1 - \int_0^1 n x^{n-1} \frac{e^{2x}}{2} \, \mathrm{d}x = \frac{e^2}{2} - \frac{n}{2} J_{n-1} \quad \text{if } n \geqslant 1$$

which is our reduction formula. We first note

$$J_0 = \int_0^1 e^{2x} \, \mathrm{d}x = \left[\frac{e^{2x}}{2} \right]_0^1 = \frac{e^2 - 1}{2},$$

and then applying the reduction formula the calculations made in Example 3 look so much easier on the eye:

$$J_3 = \frac{e^2}{2} - \frac{3}{2} J_2 = \frac{e^2}{2} - \frac{3}{2} \left(\frac{e^2}{2} - \frac{2}{2} J_1 \right) = \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3}{2} \left(\frac{e^2}{2} - \frac{1}{2} J_0 \right) = \frac{e^2}{8} + \frac{3}{8}.$$

■

Some integrands may involve two variables, such as:

Example 9 *Calculate for positive integers m, n the integral*

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, \mathrm{d}x.$$

Solution. Calculating either $B(m,1)$ or $B(1,n)$ is easy; for example

$$B(m,1) = \int_0^1 x^{m-1} \, \mathrm{d}x = \frac{1}{m}. \tag{1.1}$$

So it would seem best to find a reduction formula that moves us towards either of the integrals $B(m,1)$ or $B(1,n)$. Using integration by parts, if $n \geqslant 2$ we have

$$\begin{aligned} B(m,n) &= \left[\frac{x^m}{m} (1-x)^{n-1} \right]_0^1 - \int_0^1 \frac{x^m}{m} \times (n-1) \times (-1) (1-x)^{n-2} \, \mathrm{d}x \\ &= 0 + \frac{n-1}{m} \int_0^1 x^m (1-x)^{n-2} \, \mathrm{d}x \\ &= \frac{n-1}{m} B(m+1, n-1). \end{aligned}$$

So if $n \geqslant 2$ we can apply this to see

$$\begin{aligned} B(m,n) &= \frac{n-1}{m} B(m+1, n-1) \\ &= \frac{n-1}{m} \times \frac{n-2}{m+1} B(m+2, n-2) \\ &= \left(\frac{n-1}{m} \right) \left(\frac{n-2}{m+1} \right) \cdots \left(\frac{1}{m+n-2} \right) B(m+n-1, 1) \\ &= \left(\frac{n-1}{m} \right) \left(\frac{n-2}{m+1} \right) \cdots \left(\frac{1}{m+n-2} \right) \frac{1}{m+n-1} \\ &= \frac{(m-1)!\,(n-1)!}{(m+n-1)!}. \end{aligned}$$

Equation (1.1) shows this formula also holds for $n = 1$. ■

1.3 Appendix: Hyperbolic Functions

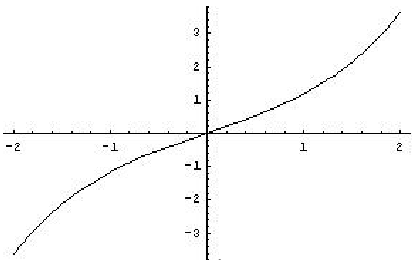
Given a real (or complex) number z then hyperbolic cosine $\cosh z$ and hyperbolic sine $\sinh z$ are defined by:

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}.$$

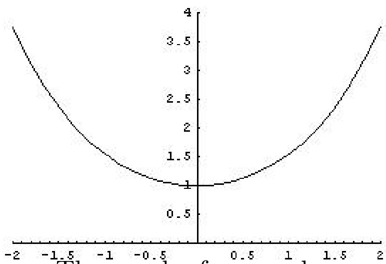
In a similar fashion to the trigonometric functions the following are also defined

$$\begin{aligned}\tanh z &= \frac{\sinh z}{\cosh z}, \qquad \coth z = \frac{\cosh z}{\sinh z}\\ \operatorname{sech} z &= \frac{1}{\cosh z}, \qquad \operatorname{csch} z = \frac{1}{\sinh z}.\end{aligned}$$

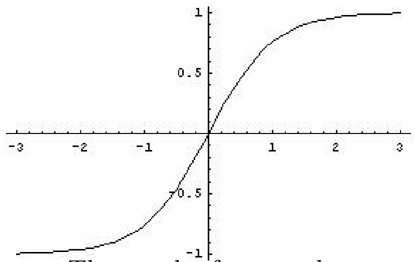
The graphs of the sinh, cosh and tanh look like



The graph of $y = \sinh x$



The graph of $y = \cosh x$



The graph of $y = \tanh x$

The functions can also easily defined by power series, which converge for all real (or complex) z.

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{z^{2n+1}}{(2n+1)!} + \cdots \qquad \text{and} \qquad \cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2n}}{(2n)!} + \cdots$$

In a similar manner to trigonometric functions the following identities hold:

$$\cosh^2 z = 1 + \sinh^2 z, \qquad \operatorname{sech}^2 z + \tanh^2 z = 1, \qquad \coth^2 z = 1 + \operatorname{csch}^2 z.$$

Consequently as t varies over the reals $(\pm\cosh t, \sinh t)$ parametrise the two branches of the hyperbola $x^2 - y^2 = 1$, which explains the name *hyperbolic* functions.

The hyperbolic functions also satisfy the following identities

$$\begin{aligned}\sinh(z+w) &= \sinh z \cosh w + \cosh z \sinh w, &\qquad \sinh 2z &= 2 \sinh z \cosh z,\\ \cosh(z+w) &= \cosh z \cosh w + \sinh z \sinh w, &\qquad \cosh 2z &= 2\cosh^2 z - 1 = 2\sinh^2 z + 1,\\ \tanh(z+w) &= \frac{\tanh w + \tanh z}{1 + \tanh w \tanh z}, &\qquad \tanh 2z &= \frac{2\tanh z}{1 + \tanh^2 z}.\end{aligned}$$

$$\begin{aligned}\sinh(iz) &= i \sin z, \qquad \sin(iz) = i \sinh z,\\ \cosh(iz) &= \cos z, \qquad \cos(iz) = \cosh z.\end{aligned}$$

The derivatives of the trigonometric functions are

$$\frac{\mathrm{d}}{\mathrm{d}z}\cosh z = \sinh z, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\sinh z = \cosh z, \qquad \frac{\mathrm{d}}{\mathrm{d}z}\tanh z = \operatorname{sech}^2 z.$$

Their inverses are given by the following formulae:

$$\begin{aligned}\sinh^{-1} x &= \ln\left(x + \sqrt{x^2+1}\right) \quad \text{for} \quad x \in \mathbb{R};\\ \cosh^{-1} x &= \ln\left(x + \sqrt{x^2-1}\right) \quad \text{for } x \geqslant 1;\\ \tanh^{-1} x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad \text{for} \quad |x| < 1.\end{aligned}$$

1.4 Exercises

Exercise 1 *Determine*

$$\int \frac{\ln x}{x}\ \mathrm{d}x, \qquad \int x\sec^2 x\ \mathrm{d}x, \qquad \int \frac{\mathrm{d}x}{4\cos x + 3\sin x}.$$

Exercise 2 *Determine*

$$\int x^6 \ln x\ \mathrm{d}x, \qquad \int \frac{\mathrm{d}x}{1+\sqrt{x}}, \qquad \int \frac{\mathrm{d}x}{\sinh x}.$$

Exercise 3 *Evaluate*

$$\int_3^\infty \frac{\mathrm{d}x}{(x-1)(x-2)}, \qquad \int_0^{\pi/2} \cos x \sqrt{\sin x}\,\mathrm{d}x, \qquad \int_0^1 \tan^{-1} x\ \mathrm{d}x.$$

Exercise 4 *Evaluate*

$$\int_2^\infty \frac{\mathrm{d}x}{x\sqrt{x-1}}, \qquad \int_0^1 \ln x\ \mathrm{d}x, \qquad \int_0^1 \frac{\mathrm{d}x}{e^x+1}.$$

Exercise 5 *Evaluate, using trigonometric and/or hyperbolic substitutions,*

$$\int_1^2 \frac{\mathrm{d}x}{\sqrt{x^2-1}}, \qquad \int \frac{\mathrm{d}x}{\sqrt{4-x^2}}, \qquad \int_2^\infty \frac{\mathrm{d}x}{(x^2-1)^{3/2}}.$$

Exercise 6 *By completing the square in the denominator, and using the substitution*

$$x = \frac{\sqrt{2}}{3}\tan\theta - \frac{1}{3}$$

determine

$$\int \frac{\mathrm{d}x}{3x^2+2x+1}..$$

Exercise 7 *Determine*

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+2x+5}}, \qquad \int_0^\infty \frac{\mathrm{d}x}{4x^2+4x+5}.$$

Exercise 8 *By completing the square in the denominator, or otherwise, show that*

$$\int_0^1 \frac{x\,\mathrm{d}x}{x^2+2x+2} = \frac{\pi}{4} + \frac{1}{2}\ln\left(\frac{5}{2}\right) - \tan^{-1} 2.$$

Exercise 9 *Determine*

$$\int \frac{x^3-4x}{x^2+1}\,\mathrm{d}x \quad \textit{and} \quad \int \frac{x^5}{x^3-1}\ \mathrm{d}x.$$

Exercise 10 *Let*

$$I_n = \int_0^{2\pi} \sin^n x\ \mathrm{d}x.$$

for $n \geqslant 0$. Use integration by parts to show that, for $n \geqslant 2$,

$$I_n = (n-1)\left(I_{n-2} - I_n\right)$$

and hence set up a reduction formula giving I_n in terms of I_{n-2}. Show by induction, or otherwise, that

$$I_{2k} = \binom{2k}{k}\frac{\pi}{2^{2k-1}}.$$

Exercise 11 *Let*

$$I_1 = \int \frac{\sin x \ \mathrm{d}x}{\sin x + \cos x}, \quad I_2 = \int \frac{\cos x \ \mathrm{d}x}{\sin x + \cos x}.$$

By considering $I_1 + I_2$ and $I_2 - I_1$ find I_1 and I_2. generalise your method to calculate

$$\int \frac{\sin x \ \mathrm{d}x}{a \sin x + b \cos x}, \quad \text{and} \quad \int \frac{\cos x \ \mathrm{d}x}{a \sin x + b \cos x}.$$

Exercise 12 *Use the substitution $x = \frac{1}{2}(1 - \sin\theta)$ to show that*

$$B\left(\frac{3}{2}, \frac{3}{2}\right) = \int_0^1 \sqrt{x(1-x)} \, \mathrm{d}x = \frac{\pi}{8}.$$

Exercise 13 *Recall that $2\cos\theta = e^{i\theta} + e^{-i\theta}$. Show that*

$$128 \cos^7 \theta = 2\cos 7\theta + 14 \cos 5\theta + 42 \cos 3\theta + 70 \cos\theta,$$

and deduce that

$$\int \cos^7 \theta \ \mathrm{d}\theta = \frac{1}{448} \sin 7\theta + \frac{7}{320} \sin 5\theta + \frac{7}{64} \sin 3\theta + \frac{35}{64} \sin\theta + \mathit{const.}$$

Exercise 14 *Show that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$ for $x \in \mathbb{R}$.*

Exercise 15 *Consider the integrals*

$$\int_\varepsilon^1 x^\alpha \ \mathrm{d}x \quad \text{and} \quad \int_1^R x^\beta \ \mathrm{d}x$$

where $0 < \varepsilon < 1 < R$ and α, β are real. For what values of α does the first integral remain bounded as ε becomes arbitrarily small? For what values of β does the second integral remain bounded as R becomes arbitrarily large?

Exercise 16 *Let $t = \tan\frac{1}{2}\theta$. Show that*

$$\sin\theta = \frac{2t}{1+t^2}, \qquad \cos\theta = \frac{1-t^2}{1+t^2}, \qquad \tan\theta = \frac{2t}{1-t^2} \quad \text{and} \quad \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{2}{1+t^2}.$$

Use the substitution $t = \tan\frac{1}{2}\theta$ to evaluate

$$\int_0^{\pi/2} \frac{\mathrm{d}\theta}{(1+\sin\theta)^2}.$$

Exercise 17 *Let*

$$I_n = \int_0^{\pi/2} x^n \sin x \ \mathrm{d}x.$$

Evaluate I_0 and I_1.

Show, using integration by parts, that

$$I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1) I_{n-2}.$$

Hence evaluate I_5 and I_6.

Exercise 18 *Let*

$$I_n = \int_0^\infty x^n e^{-x^2} \ \mathrm{d}x.$$

Show that

$$I_n = \frac{n-1}{2} I_{n-2}$$

for $n \geqslant 2$ and hence find I_5.

Given that $I_0 = \sqrt{\pi}/2$, calculate I_6.

Exercise 19 *Show that*

$$\begin{aligned}\int \cos^5 x \, \mathrm{d}x &= \frac{5}{8}\sin x + \frac{5}{48}\sin 3x + \frac{1}{80}\sin 5x + \textit{const.}\\ &= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + \textit{const.}\end{aligned}$$

Exercise 20 *Show that*

$$\int x^n (\ln x)^m \, \mathrm{d}x = \frac{x^{n+1}(\ln x)^m}{n+1} - \frac{m}{n+1}\int x^n (\ln x)^{m-1} \, \mathrm{d}x.$$

Hence find $\int x^3 (\ln x)^2 \, \mathrm{d}x$.

Exercise 21 *Show that*

$$u^4 + 1 = \left(u^2 + 1 + \sqrt{2}u\right)\left(u^2 + 1 - \sqrt{2}u\right).$$

By making the substitution $x = \tan^{-1} u^2$, *or otherwise, find*

$$\int \sqrt{\tan x} \, \mathrm{d}x.$$

Exercise 22 *Show that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1} x = \frac{1}{1-x^2}.$$

Find real numbers A, B, C, D, E, F *such that*

$$\frac{u^3}{1-u^6} = \frac{A}{1-u} + \frac{B}{1+u} + \frac{Cu+D}{u^2+u+1} + \frac{Eu+F}{u^2-u+1}$$

for all real values u. *Hence, by using the substitution* $x = \tanh^{-1} u^3$, *or otherwise, determine*

$$\int \sqrt[3]{\tanh x} \, \mathrm{d}x.$$

Exercise 23 *For* $n = 0, 1, 2, \ldots$ *define*

$$f_n(x) = \frac{1}{n!}e^{x/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^n e^{-x}\right).$$

Show that

$$\int_0^\infty f_n(x) f_m(x) \, \mathrm{d}x = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

2. DIFFERENTIAL EQUATIONS

2.1 Introduction and History

The study of differential equations (DEs) is as old as calculus itself and dates back to the time of Newton (1643-1727) and Leibniz (1646-1716). At that time most of the interest in DEs came from applications in physics and astronomy — one of Newton's greatest achievements, in his *Principia Mathematica* (1687), was to show that a force between a planet and the sun, which is inversely proportional to the square of the distance between them, would lead to an elliptical orbit.

The study of differential equations grew as increasingly varied mathematical and physical situations led to many different differential equations, and as more and more sophisticated techniques were found to solve them. Besides in astronomy, DEs began appearing naturally in other applied areas such as fluid dynamics, heat flow, vibrations on strings, in determining the curve a chain between two points will make under its own weight, and equally in pure mathematics, in finding the shortest path between two points on a surface, the surface across a given boundary of smallest area (i.e. the shape of a soap film), the largest area a curve of fixed length can bound, etc. Still today much of applied mathematics is concerned with the solution of differential equations that arise from the modelling of real world situations, whether the aim is to describe financial markets or biological systems.

In this first course we will be studying **ordinary differential equations** (ODEs) rather than **partial differential equations** (PDEs). This means that the DEs in question will involve *full* derivatives, such as $\mathrm{d}y/\mathrm{d}x$, rather than *partial* derivatives, such as $\partial y/\partial x$. The latter notation is a measure of how a function y changes with x whilst all other variables (which y depends on) are kept constant. We will meet partial derivatives in the second course (see Chapter 5 onwards).

We give here, and solve, a simple example which involves some of the key ideas of DEs; the example here is the movement of a particle P under gravity in one vertical dimension. Suppose that we write $h(t)$ for the height (in metres, say) of P over the ground at time t. If we ignore air resistance and assume gravity (denoted as g) to be constant then h satisfies the DE

$$\frac{\mathrm{d}^2 h}{\mathrm{d}t^2} = -g. \tag{2.1}$$

The (upward) **velocity** of the particle is the quantity $\mathrm{d}h/\mathrm{d}t$ — the rate of change of distance with time. The rate of change of velocity with time is called **acceleration** and is the quantity $\mathrm{d}^2h/\mathrm{d}t^2$ on the LHS of the above equation. The acceleration here is entirely due to gravity. Note the need for a minus sign here as gravity is acting downwards.

Equation (2.1) is not a difficult DE to solve; we can integrate first once,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -gt + K_1 \tag{2.2}$$

and then again

$$h(t) = \frac{-1}{2}gt^2 + K_1 t + K_2 \tag{2.3}$$

where K_1 and K_2 are constants. Currently we don't know enough about the specific motion of particle P to be able to say anything more about these constants. Note though that whatever the values of K_1 and K_2 the graph of h against t is a parabola.

Definition 10 *An* ***ordinary differential equation*** *is a equation relating a function, say y, in one variable, say x, and finitely many of its derivatives. i.e. something that can be written in the form*

$$f\left(y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2y}{\mathrm{d}x^2}, \cdots, \frac{\mathrm{d}^ky}{\mathrm{d}x^k}\right) = 0$$

for some function f and some natural number k. Here x is the ***independent variable*** *and the DE governs how the* ***dependent variable*** *y varies with x. The equation may have no or many functions $y(x)$ which satisfy it; the problem usually is to find the most general form of* ***solution*** *$y(x)$, a function which satisfies the differential equation.*

Definition 11 *The derivative $\mathrm{d}^k y/\mathrm{d}x^k$ is said to be of order k and we say that a DE has* **order** *k if it involves derivatives of order k and less.*

Example 12 *The following are types and famous examples of ordinary differential equations.*

- A **first order differential equation** is one of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

- A **kth order inhomogeneous linear differential equation** is one of the form

$$a_k(x)\frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1} y}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)\,y = f(x),$$

  and the equation is called **homogeneous** if $f(x) = 0$.

- The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y, \qquad y(0) = 1$$

  uniquely characterises the function $y = e^x$.

- The equation for simple harmonic motion is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\omega^2 y$$

  and the DE governing the swinging of a pendulum is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{l}\sin\theta.$$

- **Bessel's equation**, which relates to vibrations in a circular membrane, is

$$x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x^2 - \nu^2\right)y = 0.$$

- **Legendre's equation,** which relates to **Laplace's equation,** which we will meet in the second course, is

$$\left(1 - x^2\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + m(m+1)\,y = 0.$$

We return now to equation (2.1), which is a second order DE. In some loose sense, solving a DE of order k involves integrating k times, though not usually in such an obvious fashion as in the case of (2.1). So we would expect the solution of an order k DE to have k undetermined constants in it, and this will be the case in most of the simple examples that we look at here. However, this is not generally the case and we will see other examples where more, or fewer, than k constants are present in the solution.

The expression (2.3) is the **general solution** for the solution of the DE (2.1), that is an expression which encompasses by means of indeterminates, K_1 and K_2, all solutions of the DE. At the moment then our solution

$$h(t) = \frac{-1}{2}gt^2 + K_1 t + K_2$$

is not unique, but rather depends on two undetermined constants. And this isn't unreasonable as the particle P could follow many a path; at the moment we don't have enough information to characterise the path uniquely.

One way of filling in the missing info would be to say how high P was at $t = 0$ and how fast it was going at that point. For example, suppose P started at a height of 100m and we threw it up into the air at a speed of 10ms^{-1} — that is

$$h(0) = 100 \quad \text{and} \quad \frac{\mathrm{d}h}{\mathrm{d}t}(0) = 10. \tag{2.4}$$

Putting these values into equations (2.2) and (2.3) we'd get

$$10 = h'(0) = -g \times 0 + K_1 \quad \text{giving } K_1 = 10,$$

and

$$100 = h(0) = \frac{-1}{2} g \times 0^2 + K_1 \times 0 + K_2 \quad \text{giving } K_2 = 100.$$

So the height of P at time t has been uniquely determined and is given by

$$h(t) = 100 + 10t - \frac{1}{2} g t^2.$$

The extra bits of information given in equation (2.4) are called **initial conditions** as they relate to the situation at $t = 0$ — the DE (2.1) with the initial conditions (2.4) is called an **initial-value problem.**

Alternatively, suppose we were told that P was thrown at time $t = 0$ from a height of 100m and was subsequently one second later caught at 105m. That is

$$h(0) = 100 \quad \text{and} \quad h(1) = 105. \tag{2.5}$$

Putting these values into the general solution (2.3) gives us

$$\begin{aligned} K_2 &= 100, \\ \frac{-1}{2} g + K_1 + K_2 &= 105 \Longrightarrow K_1 = 5 + \frac{g}{2}. \end{aligned}$$

Hence

$$h(t) = \frac{-1}{2} g t^2 + \left(5 + \frac{g}{2}\right) t + 100.$$

Again we have uniquely characterised the trajectory of P by saying where the particle is at two times. The conditions (2.5) are called **boundary conditions** and the DE (2.1) with the boundary conditions (2.5) is called a **boundary-value problem.**

Having solved the earlier initial-value problem and found an equation for h, namely

$$h(t) = 100 + 10t - \frac{1}{2} g t^2$$

then we could easily answer other questions about P's behaviour such as:

- what is the greatest height P achieves? The maximum height will be a stationary value for $h(t)$ and so we need to solve the equation $h'(0) = 0$, which has solution $t = 10/g$. At this time the height is

$$h(10/g) = 100 + \frac{100}{g} - \frac{100g}{2g^2} = 100 + \frac{50}{g}.$$

- what time does P hit the ground? To solve this we see that

$$0 = h(t) = 100 + 10t - \frac{1}{2} g t^2$$

  has solutions

$$t = \frac{-10 \pm \sqrt{100 + 200g}}{-g}.$$

  One of these times is meaningless (being negative, and so before our experiment began) and so we take the other (positive) solution and see that P hits the ground at time

$$t = \frac{10 + 10\sqrt{1 + 2g}}{g}.$$

The next example is designed to show that we should not be cavalier when solving DEs.

Example 13 *Find the general solution of the DE*

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 4y. \tag{2.6}$$

Solution. Given this equation we might argue as follows — taking square roots we get

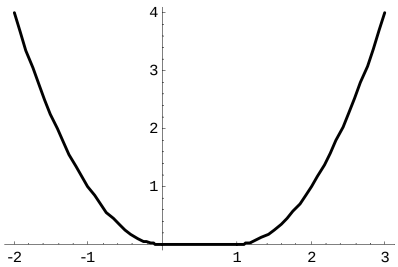
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sqrt{y} \quad \Longrightarrow \quad \frac{1}{2\sqrt{y}}\frac{\mathrm{d}y}{\mathrm{d}x} = 1. \tag{2.7}$$

From the chain rule we recognise the LHS as the derivative of $\sqrt{y}$ and so, integrating wrt x, we have $\sqrt{y} = x+K$, where K is a constant. Squaring this, we might think that the general solution has the form $y = (x+K)^2$.

What, if anything, could have gone wrong with this argument? We could have been more careful to include positive and negative square roots at the (2.7) stage, but actually we don't lose any solutions by this oversight. Thinking a little more, we might realise that we have missed the most obvious of solutions: the zero function, $y = 0$, which isn't present in our 'general' solution. At this point we might scold ourselves for dividing by zero at stage (2.7), rather than treating $y = 0$ as a separate case. But we have lost many more than just one solution at this point here by being careless. The general solution of (2.6) is in fact

$$y(x) = \begin{cases} (x-a)^2 & x \leqslant a \\ 0 & a \leqslant x \leqslant b \\ (x-b)^2 & b \leqslant x \end{cases}$$

where a and b are constants satisfying $-\infty \leqslant a \leqslant b \leqslant \infty$. We missed whole families of solutions by being careless — note also that the general solution requires *two* constants in its description even though the DE in (2.6) is only first order.



A missed solution with $a = 0$, $b = 1$.

■

2.2 Graphical Considerations

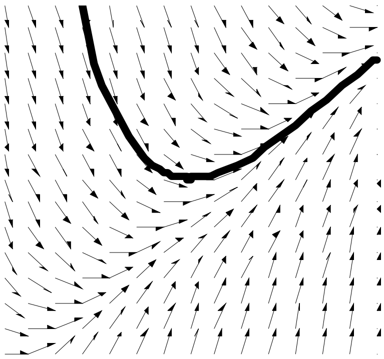
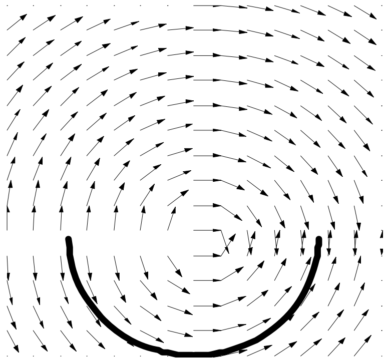
Consider the first order differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x - y.$$

By the end of §3.1 we will be able to solve such equations using integrating factors. The general solution is of the form $y = x - 1 + Ae^{-x}$. But even without this knowledge we can perform some qualitative analysis on what the solutions might look like.

In the first diagram below is a plot of the direction field of the DE — that is at each point (x, y) in the diagram is drawn a short vector (here of unit length) whose gradient equals $x - y$. Any solution of the DE is a function

$x \mapsto y(x)$ whose graph is tangential at each point to an arrow.

The direction field is plotted for $-3 < x, y < 3$ and the solution $y = x - 1 + e^{-x}$ is represented in bold. Even though we are not currently able to solve the DE we can see that solutions typically remain on one side of the line $y = x - 1$ and tend towards this line as x increases.

The second direction field is for the DE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{y+1}$$

which we will solve in the next section. The solution that is plotted is $y = -1 - \sqrt{4 - x^2}$, which is valid for $-2 < x < 2$. This is the solution to DE with initial condition $y(0) = -3$, as we will see. Again, even though we have not yet solved the DE, the solutions' semi-circular nature is clear and also that the solutions stay on either side of the line $y = -1$. As each solution approaches the line $y = -1$ we see that there is nowhere further that we can extend the solution; that is, there is no function y of x with a domain greater than $-2 < x < 2$ which satisfies the DE and initial condition.

2.3 Separable Equations

Recall that a first order differential equation has the general form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

A special class of first order differential equation contains those of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(x)\, b(y)$$

and these are known as **separable equations**. Such equations can, in principle, be easily rearranged and solved as follows:

$$\frac{1}{b(y)} \frac{\mathrm{d}y}{\mathrm{d}x} = a(x) \tag{2.8}$$

and then integrating with respect to x, we find

$$\int \frac{\mathrm{d}y}{b(y)} = \int a(x)\ \mathrm{d}x. \tag{2.9}$$

Remark 14 In some texts, or previously at A-level, equation (2.8) may be written

$$\frac{\mathrm{d}y}{b(y)} = a(x)\ \mathrm{d}x \tag{2.10}$$

and subsequently integrated. At first glance, though, this makes no rigorous sense. $\mathrm{d}y/\mathrm{d}x$ exists as the limit of an approximating gradient $\delta y/\delta x$ whereas in the limit δx and δy are both 0. So (2.10) seems to say nothing

more than $0 = 0$. However the equation can be made rigorous with knowledge of differentials (which are beyond the scope of this course) and no error will result from an equation like (2.10), but we should recognise, with our current definition of $\mathrm{d}y/\mathrm{d}x$, that the equation has no rigorous meaning.

Remark 15 Equation (2.9) may offer some practical problems as a solution to the differential equation. Firstly the functions $1/b$ and a may not be readily integrable, if at all. But further, even if they both are nicely integrable, to end with a solution of the form

$$B(y) = A(x) + \text{const.}$$

is this really solving the differential equation as asked? We originally said a solution was a function $y(x)$ which satisfies the differential equation, and our solution is not in this form, and may be difficult to put into this form.

Example 16 *(Exponential Growth and Decay) In many examples from science the rate of change of a variable is proportional to the variable itself (e.g. growth in a bacterial sample, a capacitor discharging, radioactive decay). That the rate of change with time t of a quantity, say y, is proportional to y is encoded in the differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

where k is a constant. If we separate the variables we find

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} = k$$

and integrating with respect to t we have, for some constant C,

$$\ln y = kt + C \Longrightarrow y = Ae^{kt} \tag{2.11}$$

where A is another constant. This is the general solution of the given differential equation, where A is any real number. Note that we can characterise our solution uniquely with an initial condition. In fact the exponential function $y(t) = e^t$ is characterised as the one solution of the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y, \qquad y(0) = 1.$$

Remark 17 This presentation of the solution is rather "A-level". Two points have been overlooked. Firstly what if $y = 0$? Well, we can deal with the $y = 0$ case by treating it as a separate case. Secondly though is to note that $y = Ae^{kt}$ is a solution when $A < 0$ yet here $A = e^C$ seems always to be positive. And for the remaining argument we should really write

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t} = k \Longrightarrow \ln|y| = kx + C$$

as $\ln|y|$ is an indefinite integral of the LHS whether or not y is positive. Then

$$|y| = e^C e^{kx} \Longrightarrow y = \pm e^C e^{kx}$$

gives the correct general solution.

Example 18 *Find the general solution to the separable differential equation*

$$\sin x \frac{\mathrm{d}y}{\mathrm{d}x} = y \ln y. \tag{2.12}$$

Solution. Separating variables we find

$$\frac{1}{y\ln y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sin x},$$

and integrating with respect to x we get

$$\ln|\ln y| = \ln\left|\tan\frac{x}{2}\right| + C,$$

which rearranges to

$$y = \exp\left\{A\tan\left(\frac{x}{2}\right)\right\} \tag{2.13}$$

where A is another constant. The solution is valid in the range $-\pi < x < \pi$ (or any similar domain between discontinuities of tan.). ■

Example 19 *Find the solution of the initial value problem*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-x}{y+1}, \qquad y(0) = -3. \tag{2.14}$$

Solution. The equation is separable and so we may rearrange it to

$$(y+1)\frac{\mathrm{d}y}{\mathrm{d}x} = -x$$

and integrate with respect to x to find

$$\frac{1}{2}(y+1)^2 = -\frac{1}{2}x^2 + C.$$

As $y = -3$ when $x = 0$ then $C = 2$ and we have

$$x^2 + (y+1)^2 = 4, \tag{2.15}$$

which is the equation of a circle. By definition, though, a solution of a differential equation is a function y of x whereas for each x in the range $-2 < x < 2$ there are two y which satisfy (2.15). But any solution $y(x)$ of (2.14) must satisfy (2.15), so that we can solve for y obtaining

$$y = -1 - \sqrt{4 - x^2}.$$

Note we need to take the negative root as $y(0) = -3$. Note also that the solution is valid in the range $-2 < x < 2$. ■

Example 20 *Find the solution of the initial value problem*

$$\left(5y^4 - 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 1, \qquad y(0) = 1.$$

Solution. We can straight away integrate both sides wrt x and obtain

$$y^5 - y = x + C.$$

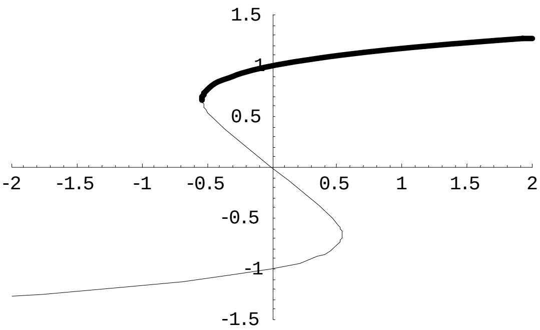
Putting $x = 0$ and $y = 1$ we see that $C = 0$ and so

$$y^5 - y = x. \tag{2.16}$$

This equation does not give us a simple expression for the solution y in terms of x, but the solution y has to satisfy equation (2.16).

If we sketch the curve $y^5 - y = x$ (see below) we see that the solution y, which goes through the point $(0, 1)$, can be extended arbitrarily to greater x but can only be extended down to $\left(\frac{-4}{5\sqrt[4]{5}}, \frac{1}{\sqrt[4]{5}}\right)$ where $\mathrm{d}y/\mathrm{d}x$ becomes infinite — there are no values we can assign to y for $x < \frac{-4}{5\sqrt[4]{5}}$ without having a discontinuity in the solution y.

On the other hand, we cannot solve equation (2.16) exactly for given x and it remains our only way to describe the solution. It should be appreciated though that the whole curve is not a solution but just the part which appears in bold on the graph below.



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2.4 Equations Adjustable to Separable Equations

By a **homogeneous polar** differential equation we will mean one of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right). \tag{2.17}$$

- **Note these DEs are often simply called homogeneous, but we will use the term homogeneous polar here to distinguish them from other DEs we will meet later and refer to as homogeneous.**

These can be solved with a substitution of the form

$$y(x) = v(x)\,x \tag{2.18}$$

to get an equation in terms of x and the new dependent variable v. Note from the product rule of differentiation that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

and so making the substitution (2.18) into the DE (2.17) gives us the new DE

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = f(v) - v,$$

which is a separable DE.

Example 21 *Find the general solution of the DE*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+4y}{2x+3y}.$$

Solution. At first glance this may not look like a homogeneous polar DE, but we can see this to be the case by rewriting the RHS as

$$\frac{1+4\frac{y}{x}}{2+3\frac{y}{x}}.$$

If we make the substitution $y(x) = xv(x)$ then we have

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+4v}{2+3v}.$$

Rearranging this gives

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+4v}{2+3v} - v = \frac{1+2v-3v^2}{2+3v}$$

and so, separating the variables, we find

$$\int \frac{2+3v}{1+2v-3v^2}\,\mathrm{d}v = \int \frac{\mathrm{d}x}{x}.$$

Using partial fractions gives

$$= \frac{1}{4}\int \left(\frac{5}{1-v} + \frac{3}{1+3v}\right)\,\mathrm{d}v = \ln|x| + C,$$

and resubstituting $v = y/x$ leads us to the general solution

$$-\frac{5}{4}\ln\left|1-\frac{y}{x}\right| + \frac{1}{4}\ln\left|1+\frac{3y}{x}\right| = \ln|x| + C.$$

This can be simplified somewhat to

$$x + 3y = A(x-y)^5$$

where A is a constant. ■

Example 22 *Solve the initial value problem*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x+y+2}, \qquad y(0) = 1. \tag{2.19}$$

Solution. This DE is not homogeneous polar, but it can easily be made into such a DE with a suitable change of variables. We introduce new variables

$$X = x + a \quad \text{and} \quad Y = y + b,$$

with the aim to choose a and b so that the DE becomes homogeneous polar. If we make these substitutions then the RHS equals

$$\frac{Y-b}{X+Y+2-a-b}$$

which is homogeneous polar if $b = 0$ and $2 - a - b = 0$ i.e. $a = 2$. With these values of a and b, noting further that $\mathrm{d}Y/\mathrm{d}X = \mathrm{d}y/\mathrm{d}x$ and that the initial condition has become $1 = y(x=0) = Y(x=0) = Y(X=2)$, our initial value problem now reads as

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{Y}{X+Y}, \quad Y(2) = 1.$$

Substituting in $Y = VX$ gives us

$$V + X\frac{\mathrm{d}V}{\mathrm{d}X} = \frac{VX}{X+VX} = \frac{V}{1+V}, \qquad V(2) = \frac{1}{2}.$$

Rearranging and separating the variables gives

$$\frac{1}{V} - \ln|V| = \int \left(-\frac{1}{V^2} - \frac{1}{V}\right) \mathrm{d}V = \int \frac{\mathrm{d}X}{X} = \ln|X| + K.$$

Substituting in our initial condition, $V(2) = 1/2$, we see that $K = 2$ and so

$$\frac{1}{V} - \ln V = \ln X + 2, \qquad X, V > 0$$

becomes, when we remember $V = Y/X$, and with some rearranging

$$X - Y\ln Y = 2Y \qquad X, Y > 0.$$

Further, as $X = x + 2$ and $Y = y$, our solution to the initial value problem (2.19) has become

$$x + 2 = 2y + y\ln y \qquad x > -2, y > 0.$$

■

Example 23 *By means of a substitution transform the following into a separable equation and find its general solution:*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos(x+y).$$

Solution. This is neither separable, nor homogeneous polar, but the substitution

$$z = x + y$$

would seem a sensible one to simplify the RHS. We might then hope to get a separable equation in x and z. As $y = z - x$ then

$$\frac{\mathrm{d}z}{\mathrm{d}x} - 1 = \cos z.$$

This is separable and we find

$$\int \frac{\mathrm{d}z}{\cos z + 1} = x + C.$$

Integrating the LHS is made easier with the trigonometric identity $\cos z + 1 = 2\cos^2(z/2)$, giving

$$\tan\frac{z}{2} = \frac{1}{2}\int \sec^2\left(\frac{z}{2}\right)\,\mathrm{d}z = x + C.$$

So

$$\tan\left(\frac{x+y}{2}\right) = x + C$$

and rearranging gives

$$y = 2\tan^{-1}(x + C) - x, \qquad x \in \mathbb{R}$$

as the general solution. ■

2.5 Exercises

Exercise 24 *Which of these solutions in (2.13) satisfy the initial condition $y(0) = 1$? How many solutions satisfy the initial condition $y(0) = 2$? Why are these answers unsurprising when we look at the original differential equation (2.12)?*

Exercise 25 *Find the solutions of the initial-value problems:*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xe^x, \qquad y(0) = 0; \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}x} = -(x+1)\,y^3, \qquad y(0) = \frac{1}{2}.$$

Exercise 26 *Find the general solutions of the following separable differential equations.*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y}; \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos^2 x}{\cos^2 2y}; \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = e^{x+2y}.$$

Exercise 27 *Find all solutions of the following separable differential equations:*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y - xy}{xy - x}; \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin^{-1} x}{y^2\sqrt{1 - x^2}}.$$

Exercise 28 *Let $x(t)$ be the solution of the initial-value problem*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (1 + x)\,e^x, \quad x(0) = 0,$$

and let T be the value of t for which $x(T) = 1$. Express T as an integral and show that

$$\frac{1}{2} - \frac{1}{2e} < T < 1 - \frac{1}{e}$$

Exercise 29 *By making a substitution $z = \mathrm{d}y/\mathrm{d}x$, or otherwise, solve the initial-value problem*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \left(1 + 3x^2\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \quad \text{where } y(1) = 0 \ \text{ and } \ y'(1) = \frac{-1}{2}.$$

Exercise 30 *Find the solution of the following initial-value problems. On separate axes sketch the solution to each problem.*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - 2x}{y}, \quad y(1) = -2; \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x\left(x^2 + 1\right)}{4y^3}, \quad y(0) = \frac{-1}{\sqrt{2}}; \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 + y^2}{1 + x^2}, \quad y(0) = 1.$$

Exercise 31 *Consider the initial-value problem*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{k^2}{y} - 1}, \qquad y(0) = 0,$$

where k is a positive constant.
(a) By introducing the change of variable $y = k^2 \sin^2(\theta/2)$, or otherwise, show that

$$x(\theta) = \frac{k^2}{2}(\theta - \sin\theta), \qquad y(\theta) = \frac{k^2}{2}(1 - \cos\theta).$$

(b) Sketch the curve for $0 \leqslant \theta \leqslant 2\pi$. Show that, in this range, the curve has arc-length $4k^2$ and bounds (with the x-axis) a region with area $3\pi k^4/4$.

Exercise 32 *Find two differentiable functions $y : \mathbb{R} \to \mathbb{R}$ which solve the initial value problem*

$$\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| = |y|, \qquad y(0) = 1.$$

Exercise 33 *Verify that*

$$y(x) = \begin{cases} \frac{1}{2}e^x - 1 & \text{for } x \leqslant \ln 2 \\ 1 - 2e^{-x} & \text{for } \ln 2 \leqslant x \end{cases}$$

is a solution of the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - |y|, \qquad y(0) = \frac{-1}{2}.$$

Sketch a graph of the solution.

Exercise 34 *Find the solution of the initial-value problem*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{1}{1+x^2}, \qquad y(0) = y'(0) = 0,$$

and confirm that $y(1) = \frac{\pi}{4} - \frac{1}{2}\ln 2$.

Exercise 35 *By making the substitution $y(x) = xv(x)$ in the following homogeneous polar equations, convert them into separable differential equations involving v and x, which you should then solve.*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2+y^2}{xy}; \qquad x\frac{\mathrm{d}y}{\mathrm{d}x} = y + \sqrt{x^2+y^2}.$$

Exercise 36 *Solve the initial-value problems*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy^2+x}{x^2y-y}, \quad y\left(\sqrt{2}\right) = 0; \qquad \sin x \sin y \frac{\mathrm{d}y}{\mathrm{d}x} = \cos x \cos y, \quad y\left(\frac{\pi}{2}\right) = \pi.$$

Exercise 37 *Solve the equation*

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} + xy = y^2.$$

Exercise 38 *By introducing the new dependent variable $z = y^2$ solve the equation*

$$\left(y^4 - 3x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} + xy = 0.$$

Exercise 39 *Make substitutions of the form $x = X + a$, $y = Y + b$, to turn the differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y-3}{x-y-1}$$

into a homogeneous polar differential equation in X and Y. Hence find the general solution of the above equation.

Exercise 40 *Show that the differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+y-1}{2x+2y-1}$$

cannot be transformed into a homogeneous polar differential equation by means of substitutions $x = X + a$, $y = Y + b$.

By means of the substitution $z = x + y$ *find the general solution of the equation.*

Exercise 41 *A particle* P *moves in the* xy*-plane. Its co-ordinates* $x(t)$ *and* $y(t)$ *satisfy the equations*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + y \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = x - y,$$

and at time $t = 0$ *the particle is at* $(1, 0)$. *Find, and solve, a homogeneous polar equation relating* x *and* y.

By changing to polar co-ordinates, or otherwise, sketch the particle's journey for $t \geqslant 0$.

Exercise 42 *Find the general solutions of the differential equations*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sin^2(x - 2y); \qquad x\frac{\mathrm{d}y}{\mathrm{d}x} + y = e^{xy}.$$

Exercise 43 *Find the general solutions of the differential equations*

$$\sin^2 x \frac{\mathrm{d}y}{\mathrm{d}x} + \left(\sin^2 x + (x+y)\sin 2x\right) = 0; \qquad e^y \frac{\mathrm{d}y}{\mathrm{d}x} = x + e^y - 1.$$

Exercise 44 *Find the general solutions of the differential equations*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (x + y - 4)^2; \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{2x + 3y - 2}.$$

Exercise 45 *Find the solutions of the initial-value problems consisting of the differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - |y|$$

and the initial conditions (a) $y(0) = 2$, *(b)* $y(0) = \frac{1}{2}$, *(c)* $y(0) = -2$. *Sketch these three solutions on the same axes.*

Exercise 46 *If* $u = 1 + \tan y$ *calculate* $\mathrm{d}(\ln u)/\mathrm{d}y$. *Hence find the general solution of*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan x \cos y(\cos y + \sin y).$$

Exercise 47 *Find a complete solution of*

$$x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - y\frac{\mathrm{d}y}{\mathrm{d}x} + A = 0$$

where A *is a positive constant.*

Exercise 48 * *Write the left hand side of the differential equation*

$$(2x + y) + (x + 2y)\frac{dy}{dx} = 0,$$

in the form

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(x, y)) = 0,$$

where $F(x, y)$ *is a polynomial in* x *and* y. *Hence find the general solution of the equation.*

Use this method to find the general solution of

$$(y\cos x + 2xe^y) + \left(\sin x + x^2 e^y - 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

3. LINEAR DIFFERENTIAL EQUATIONS

An important class of differential equations consists of the linear differential equations. These are important because of their theory, because a great many important ODEs are linear, and also because a linear differential equation can sometimes be used to approximate a non-linear equation.

Definition 24 *An* ***inhomogeneous linear DE*** *of order k is one of the form*

$$a_k(x)\frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1} y}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)\,y = f(x)$$

where $a_k(x) \neq 0$. If $f(x) = 0$ then the DE is called ***homogeneous linear****.*

3.1 Integrating Factors

A first order homogeneous linear differential equation is one of the form

$$P(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)\,y = 0$$

and we can see that this DE is also separable — we have already tackled such DEs. A first order inhomogeneous linear DE is of the form

$$P(x)\frac{\mathrm{d}y}{\mathrm{d}x} + Q(x)\,y = R(x)\,. \tag{3.1}$$

and these can be approached by using *integrating factors.*

- **The idea is to multiply the LHS of (3.1) by a function $I(x)$ of x which will make it the derivative of a product of the form $A(x)\,y$.**

Consider the first order inhomogeneous linear DE (3.1). In general, the LHS of (3.1) won't be expressible as the derivative of a product $A(x)\,y$. However, if we multiply both sides of the DE by an appropriate **Integrating Factor** $I(x)$ then we can turn the LHS into the derivative such a product. Let's first of all simplify the equation by dividing through by $P(x)$, and then multiply by an integrating factor $I(x)$ (which we have yet to determine) to get

$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + I(x)\frac{Q(x)}{P(x)}y = I(x)\frac{R(x)}{P(x)}. \tag{3.2}$$

We would like the LHS to be the derivative of a product $A(x)\,y$ for some function $A(x)$; from the product rule $A(x)\,y$ differentiates to

$$A(x)\frac{\mathrm{d}y}{\mathrm{d}x} + A'(x)\,y. \tag{3.3}$$

So, equating the coefficients of y and y' in (3.2) and (3.3), we have

$$A(x) = I(x) \quad \text{and } A'(x) = \frac{I(x)\,Q(x)}{P(x)}.$$

Rearranging this gives

$$\frac{I'(x)}{I(x)} = \frac{Q(x)}{P(x)}.$$

The LHS is the derivative of $\ln I(x)$ and so we see

$$I(x) = \exp\int \frac{Q(x)}{P(x)}\,\mathrm{d}x.$$

(We are only looking for one such $I(x)$ with this property; we do not need to worry about the constant of integration.) For this $I(x)$ (3.2) now reads as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(I(x)\,y\right) = \frac{I(x)\,R(x)}{P(x)}$$

which has the general solution

$$y(x) = \frac{1}{I(x)}\left(\int \frac{I(x)\,R(x)}{P(x)}\,\mathrm{d}x + \text{const.}\right).$$

Example 25 *Find the general solution of the DE*

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + (x-1)\,y = x^2.$$

Solution. If we divide through by x we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(1 - \frac{1}{x}\right) y = x$$

and we see that the integrating factor is

$$I(x) = \exp\int\left(1 - \frac{1}{x}\right)\,\mathrm{d}x = \exp\left(x - \ln x\right) = \frac{1}{x}e^x.$$

Multiplying through by the integrating factor gives

$$\frac{1}{x}e^x\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{1}{x} - \frac{1}{x^2}\right)e^x y = e^x,$$

which, by design, rearranges to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}e^x y\right) = e^x.$$

Integrating gives

$$\frac{1}{x}e^x y = e^x + K$$

where K is a constant, and rearranging gives

$$y(x) = x + Kxe^{-x}$$

as our general solution. ■

Example 26 *Solve the initial value problem*

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = 1, \quad y(0) = 0.$$

Solution. The integrating factor here is

$$I(x) = \exp\int 2x\,\mathrm{d}x = \exp\left(x^2\right).$$

Multiplying through we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2}y\right) = e^{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + 2xe^{x^2}y = e^{x^2}.$$

Noting that $y(0) = 0$, when we integrate this we arrive at

$$e^{x^2}y = \int_0^x e^{t^2}\,\mathrm{d}t,$$

and rearranging gives

$$y(x) = e^{-x^2}\int_0^x e^{t^2}\,\mathrm{d}t.$$

The integral of e^{x^2} can't be expressed in a closed form involving elementary functions (hence the need for normal distribution tables etc.) and we have to leave the answer in the given form. ■

Example 27 *Solve the initial value problem*

$$y\frac{\mathrm{d}y}{\mathrm{d}x}+\sin x=y^2, \qquad y\left(0\right)=1.$$

Solution. This DE is neither linear nor separable. However if we note that

$$y\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right)$$

then we see that the substitution $z=y^2$ turns the given DE into

$$\frac{\mathrm{d}z}{\mathrm{d}x}-2z=-2\sin x, \qquad z\left(0\right)=1^2=1$$

which is solvable by integrating factors. In this case the integrating factor is e^{-2x} and we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ze^{-2x}\right)=-2e^{-2x}\sin x.$$

Integrating the RHS by parts (in a similar fashion to Example 6) we get

$$ze^{-2x}=\frac{e^{-2x}}{5}\left(4\sin x+2\cos x\right)+C.$$

As $z=1$ when $x=0$ then $C=3/5$ and so, recalling that $z=y^2$, we have

$$y=\sqrt{\frac{4\sin x+2\cos x+3e^{2x}}{5}},$$

being sure to take the positive root as $y\left(0\right)=1>0$. The solution is valid on the interval containing 0 for which $4\sin x+2\cos x+3e^{2x}>0$. ■

3.2 Second Order Homogeneous Linear Differential Equations

A second order homogeneous linear differential equation is of the form

$$P\left(x\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+Q\left(x\right)\frac{\mathrm{d}y}{\mathrm{d}x}+R\left(x\right)y=0. \tag{3.4}$$

We shall first consider the situation where, either by inspection or other means, we already know of a solution $Y\left(x\right)$.

- **The idea is to make the substitution**

$$y\left(x\right)=Y\left(x\right)z\left(x\right)$$

which transforms equation (3.4) into a first order DE involving z and x.

Note

$$\frac{\mathrm{d}y}{\mathrm{d}x}=Y\frac{\mathrm{d}z}{\mathrm{d}x}+\frac{\mathrm{d}Y}{\mathrm{d}x}z, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=Y\frac{\mathrm{d}^2z}{\mathrm{d}x^2}+2\frac{\mathrm{d}Y}{\mathrm{d}x}\frac{\mathrm{d}z}{\mathrm{d}x}+\frac{\mathrm{d}^2Y}{\mathrm{d}x^2}z.$$

Substituting these expressions into (3.4) we find

$$P\left(Y\frac{\mathrm{d}^2z}{\mathrm{d}x^2}+2\frac{\mathrm{d}Y}{\mathrm{d}x}\frac{\mathrm{d}z}{\mathrm{d}x}+\frac{\mathrm{d}^2Y}{\mathrm{d}x^2}z\right)+Q\left(Y\frac{\mathrm{d}z}{\mathrm{d}x}+\frac{\mathrm{d}Y}{\mathrm{d}x}z\right)+RYz=0$$

which rearranges to

$$PY\frac{\mathrm{d}^2z}{\mathrm{d}x^2}+\left(2P\frac{\mathrm{d}Y}{\mathrm{d}x}+QY\right)\frac{\mathrm{d}z}{\mathrm{d}x}+\left(P\frac{\mathrm{d}^2Y}{\mathrm{d}x^2}+Q\frac{\mathrm{d}Y}{\mathrm{d}x}+RY\right)z=0. \tag{3.5}$$

Now the bracket

$$P\frac{\mathrm{d}^2Y}{\mathrm{d}x^2}+Q\frac{\mathrm{d}Y}{\mathrm{d}x}+RY$$

equals zero as we know that Y is a solution to (3.4). Further if we let $w(x)=\mathrm{d}z/\mathrm{d}x$ then we can see (3.5) is a first order separable DE in w:

$$PY\frac{\mathrm{d}w}{\mathrm{d}x}+\left(2P\frac{\mathrm{d}Y}{\mathrm{d}x}+QY\right)w=0 \tag{3.6}$$

which is a separable equation in solvable to find w, which we may integrate to find z and so y.

Example 28 *Show that $u(x)=1/x$ is a solution of*

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2(1-x)\frac{\mathrm{d}y}{\mathrm{d}x}-2y=0.$$

Hence find the equation's general solution.

Solution. It is easy to check that $Y(x)=1/x$ is a solution as

$$x\left(\frac{2}{x^3}\right)+2(1-x)\left(\frac{-1}{x^2}\right)-2\left(\frac{1}{x}\right)=0.$$

For this example equation (3.6) reads as

$$x\times\left(\frac{1}{x}\right)\frac{\mathrm{d}w}{\mathrm{d}x}+\left(2x\left(\frac{-1}{x^2}\right)+2(1-x)\left(\frac{1}{x}\right)\right)w=0$$

where $w=\mathrm{d}z/\mathrm{d}x$ and $z(x)/x=y(x)$. Simplifying we have

$$\frac{\mathrm{d}w}{\mathrm{d}x}+(-2)w=0,$$

and separating variables gives

$$\frac{1}{w}\frac{\mathrm{d}w}{\mathrm{d}x}=2,$$

which we know from Example 16 to have general solution

$$w=Ae^{2x}.$$

Integrating we have

$$z=ae^{2x}+b$$

(where $a=A/2$) and hence

$$y=zY=\frac{ae^{2x}+b}{x}$$

is the original DE's general solution. ■

Example 29 *Show that Legendre's equation (see Example 12) with $m=1$*

$$\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}+2y=0$$

has a solution of the form $Y(x)=ax+b$ and hence determine its general solution.

Solution. The function $Y(x) = ax + b$ is a solution of the DE if

$$\left(1 - x^2\right)(0) - 2x(a) + 2(ax + b) = 0.$$

Equating coefficients we see that

$$-2a + 2a = 0; \qquad 2b = 0.$$

These show $b = 0$ and that a can be any real; in particular $Y(x) = x$ is a solution. For this example, equation (3.6) reads as

$$\left(1 - x^2\right)(x)\frac{\mathrm{d}w}{\mathrm{d}x} + \left(\left(2 - 2x^2\right)(1) + (-2x)(x)\right)w = 0,$$

where $w = \mathrm{d}z/\mathrm{d}x$ and $z = y/x$. The above simplifies to

$$\left(x - x^3\right)\frac{\mathrm{d}w}{\mathrm{d}x} + \left(2 - 4x^2\right)w = 0.$$

Separating variables we find

$$\int \frac{\mathrm{d}w}{w} = \int \frac{2 - 4x^2}{x^3 - x}\,\mathrm{d}x = \int \left(\frac{-2}{x} + \frac{-1}{x-1} + \frac{-1}{x+1}\right)\mathrm{d}x,$$

giving

$$\ln|w| = -2\ln|x| - \ln|x-1| - \ln|x+1| + C,$$

and

$$w = \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{A}{x^2(x-1)(x+1)} = A\left(\frac{-1}{x^2} + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}\right).$$

Hence the general solution is

$$y(x) = xz(x) = A + \frac{Ax}{2}\ln\left|\frac{x-1}{x+1}\right| + Bx.$$

■

3.3 The Linear Algebra behind Linear Differential Equations

Recall that a homogeneous linear DE of order k is one of the form

$$a_k(x)\frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1}y}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y = 0.$$

As you will meet in other areas of mathematics, especially in the linear algebra courses, the space of solutions has some nice algebraic properties.

Theorem 30 *Let y_1 and y_2 be solutions of a homogeneous linear differential equation and α_1, α_2 be real numbers. Then $\alpha_1 y_1 + \alpha_2 y_2$ is also a solution of the DE. Note also that the zero function is always a solution. This means that the space of solutions of the DE is a* ***real vector space****.*

Proof. We know that

$$a_k(x)\frac{\mathrm{d}^k y_1}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1}y_1}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}y_1}{\mathrm{d}x} + a_0(x)y_1 = 0, \tag{3.7}$$

$$a_k(x)\frac{\mathrm{d}^k y_2}{\mathrm{d}x^k} + a_{k-1}(x)\frac{\mathrm{d}^{k-1}y_2}{\mathrm{d}x^{k-1}} + \cdots + a_1(x)\frac{\mathrm{d}y_2}{\mathrm{d}x} + a_0(x)y_2 = 0. \tag{3.8}$$

If we add α_1 times equation (3.7) to α_2 times equation (3.8) and rearrange we find

$$a_k(x)\frac{\mathrm{d}^k(\alpha_1 y_1 + \alpha_2 y_2)}{\mathrm{d}x^k} + \cdots + a_1(x)\frac{\mathrm{d}(\alpha_1 y_1 + \alpha_2 y_2)}{\mathrm{d}x} + a_0(x)(\alpha_1 y_1 + \alpha_2 y_2) = 0,$$

which shows that $\alpha_1 y_1 + \alpha_2 y_2$ is also a solution of the DE. ■

Remark 31 The fact that all the above holds relies on nothing more than the rules

$$\frac{\mathrm{d}}{\mathrm{d}x}(f+g)=\frac{\mathrm{d}f}{\mathrm{d}x}+\frac{\mathrm{d}g}{\mathrm{d}x}, \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\alpha f)=\alpha\frac{\mathrm{d}f}{\mathrm{d}x}$$

for functions f, g and real numbers α. These rules say that differentiation is a **linear map**.

In the case when the DE is linear, but inhomogeneous, solving the inhomogeneous equation still strongly relates to the solution of the associated homogeneous equation.

Theorem 32 *Let $Y(x)$ be a solution, known as a* ***particular solution****, or* ***particular integral****, of the inhomogeneous linear DE*

$$a_k(x)\frac{\mathrm{d}^k y}{\mathrm{d}x^k}+a_{k-1}(x)\frac{\mathrm{d}^{k-1}y}{\mathrm{d}x^{k-1}}+\cdots+a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x}+a_0(x)\,y=f(x). \tag{3.9}$$

That is $y=Y$ satisfies the above. Then a function $y(x)$ is a solution of the inhomogeneous linear DE (3.9) if and only if $y(x)$ can be written as

$$y(x)=z(x)+Y(x)$$

where $z(x)$ is a solution of the corresponding homogeneous linear DE

$$a_k(x)\frac{\mathrm{d}^k z}{\mathrm{d}x^k}+a_{k-1}(x)\frac{\mathrm{d}^{k-1}z}{\mathrm{d}x^{k-1}}+\cdots+a_1(x)\frac{\mathrm{d}z}{\mathrm{d}x}+a_0(x)\,z=0. \tag{3.10}$$

The solution $z(x)$ to the corresponding homogeneous DE is known as the ***complementary function****.*

Proof. If $y(x)=z(x)+Y(x)$ is solution of (3.9) then

$$a_k(x)\frac{\mathrm{d}^k(Y+z)}{\mathrm{d}x^k}+a_{k-1}(x)\frac{\mathrm{d}^{k-1}(Y+z)}{\mathrm{d}x^{k-1}}+\cdots+a_1(x)\frac{\mathrm{d}(Y+z)}{\mathrm{d}x}+a_0(x)(Y+z)=f(x).$$

Rearranging the brackets we get

$$\left(a_k(x)\frac{\mathrm{d}^k z}{\mathrm{d}x^k}+\cdots+a_0(x)\,z\right)+\left(a_k(x)\frac{\mathrm{d}^k Y}{\mathrm{d}x^k}+\cdots+a_0(x)\,Y\right)=f(x).$$

Now the second bracket equals $f(x)$ as $Y(x)$ is a particular solution of (3.9). Hence the first bracket must equal zero — that is $z(x)$ is a solution of the corresponding homogeneous DE (3.10). ■

Remark 33 In practice, a particular solution is usually found by educated guess work and trial and error with functions that are roughly of the same type as $f(x)$.

Remark 34 The space of solutions of (3.9) is not a vector space. They form what is known as an *affine space*. A homogeneous linear DE always has 0 as a solution, whereas this is not the case for the inhomogeneous equation. Compare this with 3D geometry: a plane through the origin is a vector space and if vectors $\mathbf{a}$ and $\mathbf{b}$ span it then every point will have a position vector $\lambda\mathbf{a}+\mu\mathbf{b}$; points on a plane parallel to it will have position vectors $\mathbf{p}+\lambda\mathbf{a}+\mu\mathbf{b}$ where $\mathbf{p}$ is some point on the plane. The point $\mathbf{p}$ acts as a choice of origin in the plane, playing the same role as Y in the above.

Example 35 *Find the general solution of*

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2(1-x)\frac{\mathrm{d}y}{\mathrm{d}x}-2y=12x. \tag{3.11}$$

Solution. We already showed in Example 28 that the general solution of the corresponding homogeneous equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2(1-x)\frac{\mathrm{d}y}{\mathrm{d}x}-2y=0$$

is

$$y(x)=\frac{ae^{2x}+b}{x}.$$

So we just need to find a particular solution of (3.11). A reasonable first attempt would be to see if there is a solution of the form

$$Y(x) = Ax + B,$$

considering that the RHS equals $12x$ and that a function like $Ax+B$ differentiates to a similar type of function. Such a Y is a solution if

$$x \times 0 + 2(1-x)A - 2(Ax+B) = 12x.$$

Rearranging this becomes

$$(2A - 2B) - 4Ax = 12x.$$

So we see that $A = -3$ and $B = -3$. That is $Y(x) = -3x - 3$ is a particular solution. So the general solution of (3.11) is

$$y(x) = \frac{ae^{2x} + b}{x} - 3x - 3.$$

■

We will meet further examples of inhomogeneous linear DEs in Section 4.2.

3.4 Exercises

Exercise 49 *Use the method of integrating factors to solve the following equations with initial conditions*

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} + xy &= x \quad \textit{where} \quad y(0) = 0, \\ 2x^3 \frac{\mathrm{d}y}{\mathrm{d}x} - 3x^2 y &= 1 \quad \textit{where} \quad y(1) = 0, \\ \frac{\mathrm{d}y}{\mathrm{d}x} - y \tan x &= 1 \quad \textit{where} \quad y(0) = 1. \end{aligned}$$

Exercise 50 *Solve the following differential equations:*

$$\begin{aligned} \left(1 - x^2\right) \frac{\mathrm{d}y}{\mathrm{d}x} + 2xy &= \left(1 - x^2\right)^{3/2}; \\ \frac{\mathrm{d}y}{\mathrm{d}x} - (\cot x)\, y + \csc x &= 0. \end{aligned}$$

Exercise 51 *By treating y as the independent variable, solve*

$$(x + y^3) \frac{\mathrm{d}y}{\mathrm{d}x} = y.$$

Exercise 52 *Solve the following differential equations:*

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\cos y - x \tan y}; \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{e^y - x}; \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3y}{3y^{2/3} - x}. \end{aligned}$$

Exercise 53 ** Show that multiplying the equation*

$$xy \frac{\mathrm{d}y}{\mathrm{d}x} + \left(2x^2 + y + x\right) = 0$$

by the integrating factor x turns the equation into one of the form

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(F(x, y)\right) = 0$$

where $F(x, y)$ is a polynomial in x and y.

Exercise 54 *Show that* $u(x) = 1/x^2$ *is a solution of*

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (x+1)\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0.$$

Hence find the equation's general solution.

Exercise 55 *Find a particular solution of*

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x(x+2)\frac{\mathrm{d}y}{\mathrm{d}x} + (x+2)y = 0$$

and hence find the general solution.

Exercise 56 *Find a particular solution to the differential equation*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = f(x),$$

for each of the following different choices of $f(x)$*:*

$$f(x) = x^2; \qquad f(x) = e^x; \qquad f(x) = \sin x.$$

Exercise 57 *Find a particular solution to the differential equation*

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2y = 6x + \sin x.$$

Exercise 58 *Find a particular solution to the differential equation*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = f(x)$$

when (i) $f(x) = \sin^2 x$*, (ii)* $f(x) = \sin x$*, (iii)* $f(x) = x\sin 2x$*.*

Exercise 59 ** A mass* P *swinging on the end of a light rod is governed by the differential equation*

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{l}\sin\theta.$$

The pendulum starts from rest at $\theta = \alpha$*.*

(a) Let $\omega = \mathrm{d}\theta/\mathrm{d}t$*. Show that* $\mathrm{d}^2\theta/\mathrm{d}t^2 = \omega\mathrm{d}\omega/\mathrm{d}\theta$ *and deduce that*

$$\frac{1}{2}\omega^2 = \frac{g}{l}(\cos\theta - \cos\alpha).$$

(b) Hence show that the time T *of an oscillation equals*

$$T = 4\sqrt{\frac{l}{2g}}\int_0^\alpha \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\alpha}}.$$

Show that if only small oscillations are performed so that the approximation $\cos\theta \approx 1 - \theta^2/2$ *applies for* $0 \leqslant \theta \leqslant \alpha$*, then*

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Exercise 60 ** **Bernoulli's equation** has the form*

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

Show that Bernoulli's equation can be reduced to an inhomogeneous linear first order equation by means of the substitution $z = y^{1-n}$*.*

Exercise 61 ** Solve the following differential equations*

$$\frac{\mathrm{d}y}{\mathrm{d}x}+y=xy^{2/3};\qquad \frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{x}=2x^{3/2}y^{1/2};\qquad 3xy^2\frac{\mathrm{d}y}{\mathrm{d}x}+3y^3=1.$$

Exercise 62 *** ***Clairaut's equation*** *has the form*

$$y=x\frac{\mathrm{d}y}{\mathrm{d}x}+F\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right).$$

Show that, by differentiating with respect to x, the equation can be reduced to a first order differential equation in $\mathrm{d}y/\mathrm{d}x$.

Hence solve the differential equation

$$y=x\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2.$$

Exercise 63 ** Find a parametric solution of*

$$x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\frac{\mathrm{d}y}{\mathrm{d}x}-y=0$$

as follows.

(a) Write an equation for y in terms of $p=\mathrm{d}y/\mathrm{d}x$ and x, and show that

$$p=p^2+(2px+1)\frac{\mathrm{d}p}{\mathrm{d}x}.$$

(b) Using p as the independent variable, arrange the above as a linear first-order equation for x.

(c) Using an appropriate integrating factor show that

$$x=\frac{\ln p-p+c}{(1-p)^2}.$$

where c is a constant. Together with the expression for y in terms of p from part (a) we have a parametric solution $(x(p),y(p))$.

Exercise 64 ** An engineer builds himself a unicycle with a square wheel of side D, and wonders now what type of road surface he should make in order that he can ride his unicycle while remaining at a constant height. A mathematician friend of his says a road surface with the equation*

$$y=-D\cosh^2(x/D)$$

would work and gives her reasoning as follows:

Call the road surface $(x(s),y(s))$, parametrised by arc-length and without loss of generality start the middle of the base of the wheel at $(0,0)$ with the base parallel to the x-axis.

(a) Let's now say the square wheel has rolled distance s. Show that the centre of the wheel is at

$$(x(s),y(s))-s(x'(s),y'(s))+\frac{D}{2}(-y'(s),x'(s)).$$

(b) Hence show that a rider's height will remain constant if

$$y(s)-sy'(s)+\frac{D}{2}\sqrt{1-y'(s)^2}=\frac{D}{2}.$$

(c) By differentiating this equation with respect to s, or otherwise, find $y(s)$ and $x(s)$, and hence show that the mathematician's answer is correct.

4. LINEAR ODES WITH CONSTANT COEFFICIENTS

In this chapter we will look to treat the theory of solving linear differential equations

$$a_k \frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1} \frac{\mathrm{d}^{k-1} y}{\mathrm{d}x^{k-1}} + \cdots + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = f(x)$$

where the functions $a_0, a_1, \ldots, a_k$ are **constants**.

We have already seen in Theorem 32 that the difference between solving the inhomogeneous and homogeneous is in finding a particular solution, so for now we will concentrate on the homogeneous case.

Mainly we shall treat examples when the equations are second order, though the theory extends naturally to similar higher order equations. We begin with the example of **simple harmonic motion (SHM).** This is the equation describing the vibrating of a spring or the swinging of a pendulum through small oscillations. The DE governing such motions is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\omega^2 y. \tag{4.1}$$

where ω is a positive constant.

Example 36 *Show that the general solution of (4.1) is of the form*

$$y(x) = A \cos \omega x + B \sin \omega x.$$

The constant ω is the angular frequency of these oscillations, with the solutions having period $2\pi/\omega$.

Solution. We firstly set $v = \mathrm{d}y/\mathrm{d}x$. By the chain rule

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}v}{\mathrm{d}y} = v\frac{\mathrm{d}v}{\mathrm{d}y}$$

and the differential equation (4.1) becomes

$$v\frac{\mathrm{d}v}{\mathrm{d}y} = -\omega^2 y$$

which is separable. If we separate the variables and integrate we find

$$\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 y^2 + K.$$

Recalling $v = \mathrm{d}y/\mathrm{d}x$ we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{K - \omega^2 y^2}.$$

Again this is separable and we may solve this to find

$$x = \int \frac{\mathrm{d}y}{\sqrt{K - \omega^2 y^2}}.$$

We met such integrals in the very first section on Trigonometric Substitutions, a sensible one here being

$$y = \frac{\sqrt{K}}{\omega} \sin t$$

which simplifies the integral to

$$x = \int \frac{\left(\sqrt{K}/\omega\right)\cos t}{\sqrt{K}\cos t}\,\mathrm{d}t = \frac{t}{\omega} + L,$$

for some constant L. Recalling $y = \left(\sqrt{K}/\omega\right)\sin t$ we have

$$y = \frac{\sqrt{K}}{\omega}\sin\omega\left(x - L\right)$$

or alternatively

$$y = A\cos\omega x + B\sin\omega x$$

for constants A and B, by using the $\sin\left(\alpha+\beta\right)$ formula. ■

4.1 The Homogeneous Case

The SHM equation is a special case of the more general DE we are interested in, which is treated by the following theorem.

Theorem 37 *Consider the DE*

$$\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = 0 \tag{4.2}$$

where Q and R are real numbers. This has ***auxiliary equation*** *(AE)*

$$m^2 + Qm + R = 0.$$

Its general solution is:

1. *in the case when the AE has two distinct real solutions α and β:*

$$y\left(x\right) = Ae^{\alpha x} + Be^{\beta x};$$

2. *in the case when the AE has a repeated real solution α:*

$$y\left(x\right) = \left(Ax + B\right)e^{\alpha x};$$

3. *in the case when the AE has complex conjugate roots $\alpha + i\beta$ and $\alpha - i\beta$:*

$$y\left(x\right) = e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right).$$

Note that the following proof is not typically examined and only a knowledge of the above solutions is expected.

Remark 38 Note, in the previous SHM example, we tackled the case when the auxiliary equation's roots are $\pm\omega i$. I have aimed in the following to present proofs that will suit both those who are au fait with complex numbers and those who are not. Those with no or limited knowledge of complex numbers need only recognise that quadratic equations which don't have real roots may be written in the form $\left(x-\alpha\right)^2 + \beta^2 = 0$ for some real α and β. For those who have met Euler's result that

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

the form of the solution in case three will be less surprising.

Proof. Cases 1 and 2: Let's call the roots of the AE α and β, and assume for the moment that they are real roots, but not necessarily distinct. We can rewrite the original DE (4.2) as

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - (\alpha + \beta) \frac{\mathrm{d}y}{\mathrm{d}x} + \alpha\beta y = 0.$$

Firstly note that $Y(x) = e^{\beta x}$ is a solution as substituting this in gives

$$\beta^2 e^{\beta x} - (\alpha + \beta) \beta e^{\beta x} + \alpha\beta e^{\beta x} = 0.$$

We saw in Section 3.2 how knowledge of a solution can simplify a second order homogeneous linear DE — if we set $z(x) = y(x) e^{-\beta x}$ then from equation (3.6) we have

$$e^{\beta x} \frac{\mathrm{d}w}{\mathrm{d}x} + \left(2\beta e^{\beta x} + (-\alpha - \beta) e^{\beta x}\right) w = 0$$

where $w = \mathrm{d}z/\mathrm{d}x$. Rearranging somewhat we find

$$\frac{\mathrm{d}w}{\mathrm{d}x} = (\alpha - \beta) w. \tag{4.3}$$

We now have two cases to consider: when $\alpha = \beta$ and when $\alpha \neq \beta$. In the case when the roots are equal then (4.3) leads to the following line of argument

$$\begin{aligned} w(x) &= \mathrm{d}z/\mathrm{d}x = A \quad \text{(a constant)}, \\ z(x) &= Ax + B \quad (A \text{ and } B \text{ constants}), \\ y(x) &= z(x) e^{\beta x} = (Ax + B) e^{\beta x}, \end{aligned}$$

as we stated in Case 2 of the theorem.

In the case when the roots are distinct reals then (4.3) has solution from Example 16

$$w(x) = \frac{\mathrm{d}z}{\mathrm{d}x} = c_1 e^{(\alpha - \beta)x}$$

(where c_1 is a constant) and so integrating gives

$$z(x) = \frac{c_1}{\alpha - \beta} e^{(\alpha - \beta)x} + c_2$$

(where c_2 is a second constant) to finally find

$$y(x) = z(x) e^{\beta x} = \frac{c_1}{\alpha - \beta} e^{\alpha x} + c_2 e^{\beta x}.$$

Case 3: Suppose now that the roots of the equation are conjugate complex numbers $\alpha \pm i\beta$. For those who are aware of Euler's relation

$$e^{i\theta} = \cos\theta + i \sin\theta$$

then we can simply treat this case the same as Case 1. Allowing for A and B to be complex numbers then (4.2) has a general solution of the form

$$y(x) = Ae^{(\alpha + i\beta)x} + Be^{(\alpha - i\beta)x} = e^{\alpha} \left(A' \cos\beta x + B' \sin\beta x\right),$$

for constants A' and B'.

Alternatively here is a proof which does not rely on complex numbers or Euler's relation. The original DE (4.2) when the AE has roots $\alpha \pm i\beta$ is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2\alpha \frac{\mathrm{d}y}{\mathrm{d}x} + \left(\alpha^2 + \beta^2\right) y = 0. \tag{4.4}$$

We will first make the substitution

$$z(x) = y(x) e^{-\alpha x}.$$

Though it is not the case that $e^{\alpha x}$ is a solution this substitution will transform the DE into something familiar. Note

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\left(ze^{\alpha x}\right)=\frac{\mathrm{d}z}{\mathrm{d}x}e^{\alpha x}+\alpha ze^{\alpha x},\qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}^2z}{\mathrm{d}x^2}e^{\alpha x}+2\alpha\frac{\mathrm{d}z}{\mathrm{d}x}e^{\alpha x}+\alpha^2ze^{\alpha x}.$$

Hence (4.4) has become a new DE involving $z\left(x\right)$

$$\left(\frac{\mathrm{d}^2z}{\mathrm{d}x^2}e^{\alpha x}+2\alpha\frac{\mathrm{d}z}{\mathrm{d}x}e^{\alpha x}+\alpha^2ze^{\alpha x}\right)-2\alpha\left(\frac{\mathrm{d}z}{\mathrm{d}x}e^{\alpha x}+\alpha ze^{\alpha x}\right)+\left(\alpha^2+\beta^2\right)ze^{\alpha x}=0,$$

which simplifies to

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2}e^{\alpha x}+\beta^2ze^{\alpha x}=0.$$

Dividing through by $e^{\alpha x}$ gives

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2}=-\beta^2z$$

which we recognise as the DE for SHM. This has general solution $z=A\cos\beta x+B\sin\beta x$ and so we can conclude

$$y\left(x\right)=e^{\alpha x}\left(A\cos\beta x+B\sin\beta x\right)$$

as required. ■

Example 39 *Find the general solution of the differential equation*

$$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0$$

Solution. This has auxiliary equation

$$0=m^2-6m+9=\left(m-3\right)^2$$

which has a repeated root of 3. Hence the general solution is

$$y=\left(Ax+B\right)e^{3x}$$

■

Example 40 *Solve the equation*

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0,$$

with initial conditions

$$y\left(0\right)=1,\quad y'\left(0\right)=0.$$

Solution. This has auxiliary equation

$$0=m^2-3m+2=\left(m-1\right)\left(m-2\right)$$

which has roots $m=1$ and $m=2$. So the general solution of the equation is

$$y\left(x\right)=Ae^{x}+Be^{2x}.$$

Now the initial conditions imply

$$\begin{aligned}1&=y\left(0\right)=A+B,\\0&=y'\left(0\right)=A+2B.\end{aligned}$$

Hence

$$A=2\quad\text{and}\quad B=-1.$$

So the *unique* solution of this DE with initial solutions is

$$y\left(x\right)=2e^{x}-e^{2x}.$$

■

Example 41 *Find all solutions of the differential equation*

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

which satisfy the boundary conditions

$$y(0) = 0 \quad \textit{and} \quad y(\pi) = 0.$$

Solution. This has auxiliary equation

$$0 = m^2 - 2m + 2 = (m-1)^2 + 1$$

which has roots $m = 1 \pm i$. So the general solution of the equation is

$$y(x) = e^x (A\cos x + B\sin x).$$

Now the boundary conditions imply

$$\begin{aligned} 0 &= y(0) = A, \\ 0 &= y(\pi) = -e^\pi A. \end{aligned}$$

Hence $A = 0$ and there are no constraints on B. So the boundary-value problem has solutions of the form

$$y(x) = Be^x \sin x$$

for any B. ■

The theory behind the solving of homogeneous linear DEs with constant coefficients extends to all orders, not just to second order DEs, provided suitable adjustments are made.

Example 42 *Write down the general solution of the following DE*

$$\frac{d^7y}{dx^7} + \frac{d^6y}{dx^6} - \frac{d^5y}{dx^5} - 5\frac{d^4y}{dx^4} + 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 4y = 0$$

Solution. This has auxiliary equation

$$m^7 + m^6 - m^5 - 5m^4 + 4m^2 + 4m - 4 = 0.$$

With can see (with a little effort) that this factorises as

$$(m-1)^3 \left(m^2 + 2m + 2\right)^2 = 0$$

which has roots $1, -1+i$ and $-1-i$, the first being a triple root and the latter two double roots. So the general solution of the DE is

$$y(x) = \left(Ax^2 + Bx + C\right) e^x + (Dx + E)\, e^{-x} \cos x + (Fx + G)\, e^{-x} \sin x.$$

■

4.2 The Inhomogeneous Case

In the previous section we discussed homogeneous linear differential equations with constant coefficients — that is equations of the form

$$a_k \frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1}\frac{\mathrm{d}^{k-1} y}{\mathrm{d}x^{k-1}} + \cdots + a_1 \frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = 0.$$

These equations occur naturally, for example the simple harmonic motion equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \omega^2 y = 0$$

governing the oscillations of a spring freely vibrating. (Such an equation arises from Hooke's Law). However if the oscillations are being driven at another frequency Ω the equation could now look like

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \omega^2 y = A \sin \Omega t,$$

which is an inhomogeneous linear DE with constant coefficients. As has already been noted in Theorem 32:

- **The solutions $y(x)$ of an inhomogeneous linear differential equation are of the form $z(x)+Y(x)$ where $z(x)$ is a complementary function, i.e. a solution of the corresponding homogeneous equation, and $Y(x)$ is a particular solution of the inhomogeneous equation.**

The particular solution $Y(x)$ is usually found by a mixture of educated guesswork and trial and error.

Example 43 *Find the general solution of*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = x. \tag{4.5}$$

Solution. As the function on the right is $f(x) = x$ then it would seem sensible to *try* a function of the form

$$Y(x) = Ax + B,$$

where A and B are, as yet, undetermined constants. There is no presumption that such a solution exists, but this seems a sensible range of functions where we may well find a particular solution. Note that

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = A \quad \text{and} \quad \frac{\mathrm{d}^2 Y}{\mathrm{d}x^2} = 0.$$

So *if* $Y(x)$ is a solution of (4.5) then substituting it in gives

$$0 - 3A + 2(Ax + B) = x$$

and this is an identity which must hold for all values of x. So comparing the coefficients of x on both sides, and the constant coefficients,

$$\begin{aligned} 2A &= 1 \text{ giving } A = \frac{1}{2}, \\ -3A + 2B &= 0 \text{ giving } B = \frac{3}{4}. \end{aligned}$$

What this means is that

$$Y(x) = \frac{x}{2} + \frac{3}{4}$$

is a particular solution of (4.5). Having already found the *complementary function,* that is the general solution of the corresponding homogeneous DE in Example 40 to be $Ae^x + Be^{2x}$, then by Theorem 32 we know the general solution of (4.5) is

$$y(x) = Ae^x + Be^{2x} + \frac{x}{2} + \frac{3}{4},$$

for constants A and B. ■

Example 44 *Solve the initial value problem*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = e^{3x}, \qquad y(0) = y'(0) = 1.$$

Solution. From Example 39 we know that the general solution of the corresponding homogeneous equation is

$$y = (Ax + B)\, e^{3x}.$$

This means that trying neither $y = e^{3x}$ nor $y = xe^{3x}$ as a particular solution is worthwhile as substituting either of them into the LHS would both yield 0. Instead we will try a particular solution of the form $Y(x) = Ax^2e^{3x}$. For this Y we find

$$\frac{\mathrm{d}^2Y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}Y}{\mathrm{d}x} + 9Y = \left\{\left(2 + 12x + 9x^2\right) - 6\left(2x + 3x^2\right) + 9x^2\right\} Ae^{3x} = 2Ae^{3x}.$$

Hence a particular solution is $Y(x) = \frac{1}{2}x^2e^{3x}$. The general solution of the given inhomogeneous DE is

$$y(x) = \left(\frac{x^2}{2} + Ax + B\right) e^{3x}.$$

Now $y(0) = B = 1$; as

$$y'(x) = \left(x + A + \frac{3x^2}{2} + 3Ax + 3B\right) e^{3x}$$

then $y'(0) = A + 3B = 1$ and so $A = -2$. Hence the initial value problem has solution

$$y(x) = \left(\frac{x^2}{2} - 2x + 1\right) e^{3x}.$$

■

Example 45 *Find particular solutions of the following DE*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = f(x)$$

where

- $f(x) = \sin x$ — *Simply trying* $Y(x) = A\sin x$ *would do no good as* $Y'(x)$ *would contain* $\cos x$ *terms whilst* $Y(x)$ *and* $Y''(x)$ *would contain only* $\sin x$ *terms. Instead we need to try the more general* $Y(x) = A\sin x + B\cos x$.

- $f(x) = e^{3x}$ — *This causes few problems and, as we would expect, we can find a solution of the form* $Y(x) = Ae^{3x}$.

- $f(x) = e^x$ — *This is different to the previous case because we know* Ae^x *is part of the general solution to the corresponding homogeneous DE, and simply substituting in* $y = Ae^x$ *into the LHS will yield* 0. *Instead we can successfully try a solution of the form* $Y(x) = Axe^x$.

- $f(x) = xe^{2x}$ — *Again* Ae^{2x} *is part of the solution to the homogeneous DE. Also as with the previous function we can see that* Axe^{2x} *would only help us with a* e^{2x} *term on the RHS. So we need to move up a further power and try a solution of the form* $Y(x) = \left(Ax^2 + Bx\right) e^{2x}$.

- $f(x) = e^x \sin x$ — *Though this may look somewhat more complicated a particular solution of the form*

  $$Y(x) = e^x (A\sin x + B\cos x)$$

  can be found.

- $f(x) = \sin^2 x$ — *Making use of the identity* $\sin^2 x = (1 - \cos 2x)/2$ *we can see that a solution of the form*

  $$Y(x) = A + B\sin 2x + C\cos 2x$$

  will work.

4.3 Exercises

Exercise 65 *Determine exactly the particular solutions from amongst the suggested families of function in Example 45.*

Exercise 66 *Find the most general solution of the following homogeneous constant coefficient differential equations:*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-y=0;\qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4y=0;\qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3\frac{\mathrm{d}y}{\mathrm{d}x}+2y=0;\qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}+3y=0.$$

Exercise 67 *Find the most general solution of the following higher order homogeneous constant coefficient differential equations:*

$$\frac{\mathrm{d}^4y}{\mathrm{d}x^4}-y=0;\qquad \frac{\mathrm{d}^3y}{\mathrm{d}x^3}-y=0;\qquad \frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}+y=0;\qquad \frac{\mathrm{d}^4y}{\mathrm{d}x^4}+\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0.$$

Exercise 68 *Solve the following initial-value problems*

$$\text{(i)}\qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}+y=e^x\cos x\qquad y(0)=1,\ y'(0)=0;$$

$$\text{(ii)}\qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}+4y=\cos^2x\qquad y(0)=0,\ y'(0)=2.$$

Exercise 69 *Solve the initial-value problem*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}y}{\mathrm{d}x}+y=\cosh x,\qquad y(0)=y'(0)=0.$$

Exercise 70 *Solve the initial-value problem*

$$\frac{\mathrm{d}^{2007}y}{\mathrm{d}x^{2007}}-y=0\qquad y^{(k)}(0)=1 \text{ for } 0\leqslant k<2007.$$

Exercise 71 *By means of the substitution $z=\ln x$ find the general solution of the differential equation*

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+y=0.$$

Exercise 72 *Find the general solution of*

$$(x+1)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+3(x+1)\frac{\mathrm{d}y}{\mathrm{d}x}+y=x^2.$$

Exercise 73 *Find all the solutions (if any) of the following boundary-value problems:*

$$\text{(i)}\qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\pi^2y,\qquad y(0)=1, y(1)=-1;$$

$$\text{(ii)}\qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\pi^2y,\qquad y(0)=1, y(1)=-1;$$

$$\text{(iii)}\qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\pi^2y,\qquad y(0)=1, y(1)=0.$$

Exercise 74 *Show that, for any values of α and β, the function*

$$y(x)=\sin x\int_\alpha^x f(t)\cos t\,\mathrm{d}t-\cos x\int_\beta^x f(t)\sin t\,\mathrm{d}t$$

is a solution to the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+y=f(x).$$

Hence solve the boundary-value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+y=\csc x,\qquad y(0)=y(\pi/2)=0.$$

Exercise 75 *Show that the solutions of the differential equation*

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = \cos \Omega t$$

are bounded when $\Omega \neq \omega$*, but become unbounded when* $\Omega = \omega$*.*

Exercise 76 *Find the most general solution of the following inhomogeneous constant coefficient differential equations:*

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y &= x; \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = \sin x; \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y &= e^x; \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = e^{-x}. \end{aligned}$$

Exercise 77 *Find the general solutions of the differential equation*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (a+b)\frac{\mathrm{d}y}{\mathrm{d}x} + aby = e^{cx},$$

when (i) a, b, c *are distinct real numbers, (ii)* a *and* b *are distinct real numbers and* $c = a$*.*

Exercise 78 *Write down a family of trial functions* $y(x)$ *which will contain a particular solution of*

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = f(x),$$

for each of the following different choices of $f(x)$*:*
(i) $f(x) = x^2$*, (ii)* $f(x) = xe^x$*, (iii)* $f(x) = xe^{-2x}$*, (iv)* $f(x) = x^2 \sin x$*, (v)* $f(x) = \sin^3 x$*.*

Exercise 79 *By making the substitutions* $x = e^t$ *and* $y = v(t)\, e^t$*, or otherwise, solve the differential equation*

$$x^3 \frac{\mathrm{d}^2y}{\mathrm{d}x^2} - \left(x^2 + xy\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \left(y^2 + xy\right) = 0.$$

Exercise 80 ***Chebyshev's equation*** *is*

$$\left(1 - x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + n^2 y = 0$$

where n *is a non-negative integer. By making the substitution* $x = \cos\theta$*, show that Chebyshev's equation has two independent solutions*

$$T_n(x) = \cos\left(n \cos^{-1} x\right), \qquad V_n(x) = \sin\left(n \cos^{-1} x\right).$$

Show that

$$T_n(x) + iV_n(x) = \left(x + i\sqrt{1 - x^2}\right)^n$$

and deduce that $T_n(x)$ *is a polynomial of degree* n*.*

Exercise 81 ** Let* $a_0, a_1, \ldots, a_k$ *be real constants and suppose that the polynomial*

$$z^k + a_{k-1}z^{k-1} + \cdots + a_1 z + a_0 = 0$$

has k *distinct real roots* $\gamma_1, \gamma_2, \ldots, \gamma_k$*.*
(a) Show that

$$\det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_k \\ \gamma_1^2 & \gamma_2^2 & \gamma_3^2 & \cdots & \gamma_k^2 \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_1^{k-1} & \gamma_2^{k-1} & \gamma_3^{k-1} & \cdots & \gamma_k^{k-1} \end{pmatrix} = \prod_{1 \leqslant i < j \leqslant k} \left(\gamma_j - \gamma_i\right).$$

(b) Deduce, that for any $c_0, c_1, c_2, \ldots, c_{k-1}$*, the initial-value problem*

$$\frac{\mathrm{d}^k y}{\mathrm{d}x^k} + a_{k-1}\frac{\mathrm{d}^{k-1}y}{\mathrm{d}x^{k-1}} + \cdots + a_1\frac{\mathrm{d}y}{\mathrm{d}x} + a_0 y = 0, \quad y(0) = c_0, y'(0) = c_1, \ldots, y^{(k-1)}(0) = c_{k-1}$$

has a unique solution.

Exercise 82 ** Assuming that term-by-term differentiation is valid within a power series' range of convergence, show that the power series*

$$y(x) = \sum_{k=0}^{\infty} a_k x^k \tag{4.6}$$

is a solution of the initial-value problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0, \qquad y(0) = y'(0) = 1,$$

provided

$$a_{k+2} = \frac{-a_k}{(k+2)(k+1)}, \quad \text{and} \quad a_0 = a_1 = 1.$$

Hence determine the coefficients a_k. Solve the above initial-value problem and show that the solution agrees with the power series (4.6) above.

Exercise 83 ** Use the power series method of Exercise 82 to find the general solution of the differential equation*

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{2y}{(1-x)^2}$$

as a power series. For what range of x-values does your series converge?

Exercise 84 ** **Bessel's equation** reads as*

$$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + \left(x^2 - v^2\right) y = 0$$

where $v \geqslant 0$.

(a) Show that if v is an integer then

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,(n+v)!} \left(\frac{x}{2}\right)^{v+2n}$$

is a solution of Bessel's equation.

(b) By making the substitution $y = z/\sqrt{x}$, or otherwise, find the general solution to Bessel's equation when $v = 1/2$.

Exercise 85 ** The differential equation*

$$\left(1 - x^2\right) \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + n(n+1)\, y = 0$$

*is known as **Legendre's equation**. Find a recurrence relation amongst the coefficients of a power series solution $y(x) = \sum_{k=0}^{\infty} a_k x^k$.*

Show that, if n is a non-negative integer, there is a polynomial solution $P_n(x)$ of Legendre's equation of degree n.

*This polynomial is unique up to scalar multiplication and so it is conventional to further assume that $P_n(1) = 1$. These polynomials are known as **Legendre polynomials.** Find $P_n(x)$ for $0 \leqslant n \leqslant 4$.*

Exercise 86 ** Show that Legendre's equation can be rewritten as*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(1 - x^2\right) \frac{\mathrm{d}y}{\mathrm{d}x}\right] + n(n+1)\, y = 0.$$

Deduce, that for $n > m \geqslant 0$,

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, \mathrm{d}x = 0.$$

5. SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

5.1 Some Facts relating to Matrices

We introduce here some basic definitions and properties relating to matrices. We will use matrices later to work with simultaneous differential equations like

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x + 3y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 3x + 4y,$$

though the theory extends more generally to n linear differential equations with constant coefficients in n variables.

Definition 46 *A (real) $m \times n$ **matrix** is an array of real numbers arranged into m rows and n columns. This array is typically placed inside brackets and we write*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}.$$

*The notation a_{ij} denotes the **entry** in the ith row and jth column.*

Example 47 *Let*

$$A = (a_{ij}) = \begin{pmatrix} 1 & -3 & \pi \\ e^2 & 2.5 & 0 \end{pmatrix}.$$

This is a 2×3 matrix — $a_{12} = -3$ and $a_{21} = e^2$.

In the main we shall only be interested in 2×2 matrices.

Definition 48 *Let*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

*be two 2×2 matrices. These matrices can be **added** to form the sum $A + B$ and **multiplied** to form the product AB as follows.*

$$\begin{aligned} A + B &= \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}; \\ AB &= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}. \end{aligned}$$

Note, for example, that the 1st row, 2nd column entry in AB is calculated by taking the term by term products of the 1st row of A and 2nd column of B and adding these products (note that this is just the same as taking the dot product of the first row and second column) — that is

$$\begin{pmatrix} \boxed{a_{11} \;\; a_{12}} \\ a_{21} \;\; a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & \boxed{\begin{matrix} b_{12} \\ b_{22} \end{matrix}} \\ b_{21} & \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & \boxed{a_{11}b_{12} + a_{12}b_{22}} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

*If c is a real number, often referred to as a **scalar** in this context, then we can also form the **scalar multiple** cA by multiplying each entry of A by c, i.e.*

$$cA = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}.$$

Remark 49 More generally it is possible to

- add an $m_1 \times n_1$ matrix to an $m_2 \times n_2$ matrix if $m_1 = m_2$ and $n_1 = n_2$ — the (i,j)th entry in the sum is the sum of the matrices' (i,j)th entries;
- multiply an $m_1 \times n_1$ matrix by an $m_2 \times n_2$ matrix if $n_1 = m_2$ to produce an $m_1 \times n_2$ matrix product — the (i,j)th entry in the product is calculated by taking the term by term products of the ith row of the first matrix and the jth column of the second and adding them, i.e. we take the dot product of the ith row and jth column;
- multiply any $m \times n$ matrix by any real number c to form a scalar multiple — the (i,j)th entry in the scalar multiple is c times the (i,j)th entry in the matrix.

Example 50 *Let*

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 0 & 3 \\ -2 & 5 \end{pmatrix}.$$

Find $A+B, B+A, AB$ *and* BA.

Solution. Then

$$\begin{aligned}
A+B &= \begin{pmatrix} 2+0 & 1+3 \\ -1-2 & 3+5 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -3 & 8 \end{pmatrix}; \\
B+A &= \begin{pmatrix} 0+2 & 3+1 \\ -2-1 & 5+3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -3 & 8 \end{pmatrix}; \\
AB &= \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 0-2 & 6+5 \\ 0-6 & -3+15 \end{pmatrix} = \begin{pmatrix} -2 & 11 \\ 6 & 12 \end{pmatrix}; \\
BA &= \begin{pmatrix} 0 & 3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 0-3 & 0+9 \\ -4-5 & -2+15 \end{pmatrix} = \begin{pmatrix} -3 & 9 \\ -9 & 13 \end{pmatrix}. \\
3A &= \begin{pmatrix} 6 & 3 \\ -3 & 9 \end{pmatrix}; \qquad 3B = \begin{pmatrix} 0 & 9 \\ -6 & 15 \end{pmatrix} \\
3A+3B &= \begin{pmatrix} 6 & 12 \\ -9 & 24 \end{pmatrix} = 3(A+B)
\end{aligned}$$

■

Example 51 • ***Note that*** $A+B=B+A$ ***and this is generally the case; however*** $AB \neq BA$ ***in this case and we see that matrix multiplication is not generally commutative.***

- ***On the other hand matrix multiplication is always associative. That is***

$$(AB)C = A(BC)$$

for any 2×2 ***matrices*** A, B ***and*** C. *(We shall not prove this here.)*

- ***The distributive law***

$$aA + aB = a(A+B)$$

holds for all matrices A, B ***and reals*** a***, as does the other distributive law***

$$(a+b)A = aA + bA$$

for all matrices A ***and reals*** a, b.

Definition 52 *The* ***determinant*** *of a* 2×2 *matrix* A *written* $\det A$ *or* $|A|$ *is*

$$\det A = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Remark 53 Determinants can more generally defined for $n \times n$ matrices. In the 3×3 case

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}. \tag{5.1}$$

Example 54 *With A and B as in the previous example we have*

$$\begin{aligned} \det A &= \det \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} = 2 \times 3 - 1 \times (-1) = 7; \\ \det B &= \det \begin{pmatrix} 0 & 3 \\ -2 & 5 \end{pmatrix} = 0 \times 5 - 3 \times (-2) = 6; \\ \det (A+B) &= \det \begin{pmatrix} 2 & 4 \\ -3 & 8 \end{pmatrix} = 16 + 12 = 28 \neq \det A + \det B; \\ \det (AB) &= \det \begin{pmatrix} -2 & 11 \\ 6 & 12 \end{pmatrix} = -24 + 66 = 42 = \det A \det B. \end{aligned}$$

Proposition 55 *Let A and B be two 2×2 matrices. Then*

$$\det (AB) = \det A \times \det B.$$

Proof.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Then

$$\begin{aligned} \det (AB) &= \det \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} \\ &= (ae+bg)(cf+dh) - (af+bh)(ce+dg) \\ &= bgcf + aedh - bhce - afdg \\ &= (ad-bc)(eh-fg) \\ &= \det A \det B. \end{aligned}$$

■

Definition 56 *The 2×2 **identity** matrix is denoted by I (or I_2 if we wish to stress the 2×2 context) and equals*

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It has the property that $IA = A = AI$ for any 2×2 matrix A.

Definition 57 *The 2×2 **zero** matrix is denoted by 0 (or 0_2 if we wish to stress the 2×2 context) and equals*

$$0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It has the properties that $0A = 0 = A0$ and $A + 0 = A = 0 + A$ for any 2×2 matrix A.

Definition 58 *Given a 2×2 matrix A then an **inverse** matrix for A (which may or may not exist) is a 2×2 matrix B such that*

$$AB = I = BA.$$

If A has an inverse then it is unique (see below) and we denote it as A^{-1}.

Note if A^{-1} exists then

$$1 = \det I = \det \left(AA^{-1}\right) = \det A \det \left(A^{-1}\right)$$

means

$$\det A^{-1} = \frac{1}{\det A}.$$

*If A has an inverse then it is said to be **invertible**, and is said to be **singular** if it has no inverse.*

Proposition 59 *A 2×2 matrix A has an inverse if and only if $\det A \neq 0$. If $\det A \neq 0$ then a unique inverse exists and equals*

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Proof. If A has an inverse B then

$$1 = \det I = \det AB = \det A \det B$$

and so we see that $\det A \neq 0$. On the other hand if $\det A = ad - bc \neq 0$ then we see

$$\left\{ \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right\} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = I \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \left\{ \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right\} = I.$$

So an inverse exists.

Let's suppose two inverses B and C for A existed. That is there were two matrices B and C such that

$$BA = I = AB \quad \text{and} \quad CA = I = AC.$$

As matrix multiplication is associative then

$$C = IC = (BA)\,C = B\,(AC) = BI = B.$$

Hence the inverse we have found is indeed unique. ■

Remark 60 Two vectors $\mathbf{v}, \mathbf{w}$ are said to be **linearly dependent** or just **dependent** if one is a scalar multiple of the another — i.e. if they are parallel — and are said to be **linearly independent** or just **independent** otherwise.

Note that $ad = bc$ if and only if $\binom{a}{b}$ and $\binom{c}{d}$ are parallel. So a 2×2 matrix is invertible if its columns (or equivalently rows) are independent vectors.

Definition 61 *The* ***eigenvalues*** *of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are the roots of the quadratic equation*

$$\begin{aligned} \det(A - xI) &= \det \begin{pmatrix} a - x & b \\ c & d - x \end{pmatrix} \\ &= (a - x)(d - x) - bc \\ &= x^2 - (a + d)\,x + (ad - bc) = 0. \end{aligned}$$

Example 62 *Find the eigenvalues of the following matrices*

$$A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Solution.

$$\begin{aligned} \det(A - xI) &= \begin{vmatrix} 2 - x & 1 \\ 6 & 3 - x \end{vmatrix} = x^2 - 5x = x\,(x - 5)\,; \\ \det(B - xI) &= \begin{vmatrix} 2 - x & 1 \\ 0 & 2 - x \end{vmatrix} = (x - 2)^2\,; \\ \det(C - xI) &= \begin{vmatrix} -x & -1 \\ 1 & -x \end{vmatrix} = x^2 + 1. \end{aligned}$$

So we see the eigenvalues are $0, 5$ for A, $2, 2$ for B, and $\pm i$ for C. ■

Definition 63 *Given a 2×2 matrix A and a 2×1 column vector $\mathbf{v} = \binom{x}{y}$ (we can think of as a 2×1 matrix) then we can form the matrix product $A\mathbf{v}$ which is another 2×1 column vector. Recall*

$$A\mathbf{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}.$$

In this way A defines a map $\mathbb{R}^2 \to \mathbb{R}^2$ known as a ***linear map****. (cf. Michaelmas Linear Algebra course.) This means that*

$$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2$$

for vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and scalars c_1, c_2. In particular, $A\mathbf{0} = \mathbf{0}$.

Example 64 *Determine the following products.*

$$\begin{aligned}
\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 2 \\ 6 \end{pmatrix}; \\
\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 3 \end{pmatrix}; \\
\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -4 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 2 \\ -4 \end{pmatrix}; \\
\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} &= \begin{pmatrix} 5 \\ 15 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 3 \end{pmatrix}.
\end{aligned}$$

- Note that the matrix maps $\binom{1}{0}$ to its first column and $\binom{0}{1}$ to its second column; it is easy to check that this generally happens.
- Note, in the third line, that it is possible for a matrix to map a non-zero vector to the zero vector.
- Note that in the third and fourth lines the vectors are mapped to scalar multiples of themselves, and that these scalar are A's eigenvalues from Example 62.

Proposition 65 *The vector equation $A\mathbf{x} = \mathbf{0}$ has a non-zero solution $\mathbf{x} = \mathbf{v}$ if and only if $\det A = 0$.*

Proof. If $\det A \neq 0$ then we know from Proposition 59 that A has an inverse A^{-1}. So if $A\mathbf{v} = \mathbf{0}$ then

$$\mathbf{v} = A^{-1}A\mathbf{v} = A^{-1}\mathbf{0} = \mathbf{0},$$

and we see $\mathbf{0}$ is the only solution. On the other hand if $\det A = ad - bc = 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d \\ -c \end{pmatrix} = \begin{pmatrix} ad - bc \\ 0 \end{pmatrix} = \mathbf{0}.$$

So $\binom{d}{-c}$ is a non-zero solution unless $c = d = 0$ in which case $\binom{-b}{a}$ will do, unless $a = b = c = d = 0$ in which case any non-zero vector will do! ■

Corollary 66 *If $x = \lambda$ is a real eigenvalue of A, so that $\det(A - \lambda I) = 0$, then there is a non-zero vector $\mathbf{v}$ such that $(A - \lambda I)\mathbf{v} = \mathbf{0}$. That is*

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Definition 67 *A non-zero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ is called an* ***eigenvector*** *or a* ***λ-eigenvector.*** *λ is the corresponding* ***eigenvalue.***

- Note conversely if $A\mathbf{v} = \mu\mathbf{v}$ for some non-zero vector $\mathbf{v}$ then $(A - \mu I)\mathbf{x} = \mathbf{0}$ has a non-zero solution $\mathbf{x} = \mathbf{v}$ and so by the previous proposition $\det(A - \mu I) = 0$; in particular μ is an eigenvalue of A.
- Note that if $\mathbf{v}$ is an eigenvector then so is $c\mathbf{v}$ for any non-zero scalar c.

Example 68 *Find all the eigenvectors of the matrices A, B, C from Example 62.*

Solution. A has eigenvalues 0 and 5. We need to solve the vector equations

$$\begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}, \qquad \begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}.$$

The first vector equation says that

$$2x + y = 0 \quad \text{and} \quad 6x + 3y = 0.$$

Note that the equations are really the same equation (being scalar multiples of one another) which is why they have non-zero solutions. We can set $x = c_1$ and then $y = -2c_1$ Working similarly with the second case we see that A's eigenvectors are of the form

$$c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

B has repeated eigenvalue 2. We need to solve the vector equation

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$$

which has solutions

$$c \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Note that B does not have two independent eigenvectors.

C has no real eigenvalues and so no real eigenvectors. ■

Definition 69 *A 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is said to be **diagonal** if $b = c = 0$; that is it is of the form $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$*

Theorem 70 *Suppose that the 2×2 matrix A has distinct real eigenvalues λ_1 and λ_2. Let $\mathbf{v}_i$ for $i = 1, 2$ be corresponding λ_i-eigenvectors and let*

$$P = (\mathbf{v}_1 | \, \mathbf{v}_2),$$

that is P has columns $\mathbf{v}_1$ and $\mathbf{v}_2$. Then P is invertible and

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Proof. Suppose that

$$P\begin{pmatrix} x \\ y \end{pmatrix} = (\mathbf{v}_1 | \, \mathbf{v}_2)\begin{pmatrix} x \\ y \end{pmatrix} = x\mathbf{v}_1 + y\mathbf{v}_2 = \mathbf{0}. \tag{5.2}$$

Then

$$A\,(x\mathbf{v}_1 + y\mathbf{v}_2) = x\lambda_1\mathbf{v}_1 + y\lambda_2\mathbf{v}_2 = \mathbf{0}. \tag{5.3}$$

Now (5.3) minus λ_1 times (5.2) gives

$$(\lambda_2 - \lambda_1)\, y\mathbf{v}_2 = \mathbf{0}.$$

As $\lambda_1 \neq \lambda_2$ and $\mathbf{v}_2 \neq \mathbf{0}$, then $y = 0$ and hence $x = 0$. As (5.2) only has $\mathbf{0}$ as a solution then P is invertible by Proposition 65.

Finally

$$AP = A\,(\mathbf{v}_1 | \, \mathbf{v}_2) = (\lambda_1\mathbf{v}_1 | \, \lambda_2\mathbf{v}_2) = P\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and premultiplying by P^{-1} on both sides gives the required result. ■

Example 71 *Let A be the matrix given in Example 62. Determine whether there is a real matrix P, such that $P^{-1}AP$ is diagonal. Are there similar matrices P for B and/or C?*

Solution. In Example 62 we showed that A had eigenvalues 0 and 5 with corresponding eigenvectors $\binom{1}{-2}$ and $\binom{1}{3}$ (see Example 68). From the previous theorem we know then there is such a P which has these two eigenvectors as its columns. We can check this directly works here:

$$\begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 5 \\ 0 & 15 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 0 & 0 \\ 0 & 25 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$$

as required. A is said to be **diagonalisable**.

B on the other hand has eigenvalues 2 and 2. If such a P existed then it would follow that

$$P^{-1}BP = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I.$$

(Why would this be the case?) But then we'd have

$$B = P\,(2I)\,P^{-1} = 2PP^{-1} = 2I$$

and this is not the case; for B no such P exists and B is NOT diagonalisable.

Similarly for C, if such a real matrix P existed we'd have

$$P^{-1}CP = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

and this is clearly not possible with both P and C being real matrices. ■

5.2 Simultaneous Differential Equations

As promised, we shall now apply matrices to coupled differential equations, governing two functions x and y of a variable t, when the equations have the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = cx + dy, \qquad (a, b, c, d \in \mathbb{R}).$$

We already have two ways of approaching such systems, and we will meet a third way relating to the matrix theory we have recently met. We shall work with a specific example with initial values

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x + y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = 6x + 4y, \qquad x\,(0) = y\,(0) = 1.$$

METHOD 1: If we use the first equation to write $y = \mathrm{d}x/\mathrm{d}t - 3x$, and substitute this into the second equation we have

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 3\frac{\mathrm{d}x}{\mathrm{d}t} = 6x + 4\frac{\mathrm{d}x}{\mathrm{d}t} - 12x,$$

and hence

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 7\frac{\mathrm{d}x}{\mathrm{d}t} + 6x = 0.$$

with $x\,(0) = 1$ and $x'\,(0) = 3x\,(0) + y\,(0) = 4$. We met in Chapter 4 how to solve such linear DEs with constant coefficients.

METHOD 2: Alternatively we could note

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \Big/ \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3x + y}{6x + 4y}$$

and this is a homogeneous polar equation which we met previously in Section 2.4 and can be solved with a substitution of the form $y(x) = xv(x)$ to give an equation relating x and y. By substituting our expression for y back into the original equations in principle it should be possible to determine x and y in terms of t.

METHOD 3: Our third method relates to the linear algebra we met in the previous section. We can rewrite our two equations as a single differential equation in a vector $\mathbf{v} = \binom{x}{y}$, namely

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \begin{pmatrix} \mathrm{d}x/\mathrm{d}t \\ \mathrm{d}y/\mathrm{d}t \end{pmatrix} = \begin{pmatrix} 3x+y \\ 6x+4y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix} \mathbf{v}.$$

Let's write $A = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix}$. If we find A's eigenvalues, these turn out to be 1 and 6 and two corresponding independent eigenvectors are $\binom{1}{-2}$ and $\binom{1}{3}$ respectively. If we put these eigenvectors into the columns of a matrix

$$P = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \quad \text{so that} \quad P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix},$$

then we have

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = A\mathbf{v} = P \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} P^{-1}\mathbf{v}.$$

Hence, because the entries of P^{-1} are constant,

$$\frac{\mathrm{d}\left(P^{-1}\mathbf{v}\right)}{\mathrm{d}t} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \left(P^{-1}\mathbf{v}\right).$$

If we set $\binom{X}{Y} = P^{-1}\mathbf{v}$, so that X and Y are new functions in t, then we have

$$\frac{\mathrm{d}X}{\mathrm{d}t} = X \quad \text{and} \quad \frac{\mathrm{d}Y}{\mathrm{d}t} = 6Y$$

with

$$\begin{pmatrix} X(0) \\ Y(0) \end{pmatrix} = P^{-1} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/5 \\ 3/5 \end{pmatrix}.$$

Thus

$$X(t) = \frac{2}{5}e^{t} \quad \text{and} \quad Y(t) = \frac{3}{5}e^{6t}$$

and finally

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = P \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} \frac{2}{5}e^{t} \\ \frac{3}{5}e^{6t} \end{pmatrix} = \begin{pmatrix} \frac{2}{5}e^{t} + \frac{3}{5}e^{6t} \\ -\frac{4}{5}e^{t} - \frac{9}{5}e^{6t} \end{pmatrix}.$$

More generally for such systems of differential equations we have:

Theorem 72 *Two variables $x(t)$ and $y(t)$ satisfy the differential equations*

$$\frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy. \tag{5.4}$$

If the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has distinct real eigenvalues λ_1 and λ_2 with corresponding eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ then the general solution of (5.4) is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Ae^{\lambda_1 t}\mathbf{v}_1 + Be^{\lambda_2 t}\mathbf{v}_2. \tag{5.5}$$

Proof. The proof is simply a duplication of the working shown in Method 3 to the general case and so is omitted. ■

Whilst we won't go through the general details of what happens if the eigenvalues are repeated or complex we will treat two such examples below. Though it is worth noting

- **Formula (5.5) still holds when the eigenvalues λ_1 and λ_2 are complex provided we allow the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ to be complex and also the constants A and B. The solution can then be written as an overtly real function using the identity**

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Example 73 *Consider the linear system*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2x + y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -4x + 6y, \qquad x(0) = y(0) = 1.$$

Solution. The matrix A of the system is

$$A = \begin{pmatrix} 2 & 1 \\ -4 & 6 \end{pmatrix}$$

and has repeated eigenvalue 4. We now make the substitutions

$$X(t) = x(t)\,e^{-4t}, \qquad Y(t) = y(t)\,e^{-4t}.$$

(More generally this would read $X(t) = x(t)\,e^{-\lambda t}$, etc. when λ is the repeated eigenvalue.) Then

$$\dot{X}(t) = (\dot{x} - 4x)\,e^{-4t} = (-2x + y)\,e^{-4t} = -2X + Y, \tag{5.6}$$

$$\dot{Y}(t) = (\dot{y} - 4y)\,e^{-4t} = (-4x + 2y)\,e^{-4t} = -4X + 2Y. \tag{5.7}$$

We can see that $2\dot{X} - \dot{Y} = 0$ (generally $\dot{X}$ and $\dot{Y}$ will be proportional) and hence

$$2X - Y = \text{const.} = 2X(0) - Y(0) = 2 - 1 = 1.$$

Hence, substituting this back into (5.6) and (5.7)

$$\dot{X} = -1 \text{ giving } X = -t + c_1, \qquad \dot{Y} = -2 \text{ giving } Y = -2t + c_2.$$

As $X(0) = Y(0) = 1$ then $c_1 = c_2 = 1$ and we have our solution:

$$x(t) = (1 - t)\,e^{4t}, \qquad y(t) = (1 - 2t)\,e^{4t}.$$

■

Example 74 *Consider the linear system*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -x + y, \qquad x(0) = y(0) = 1.$$

Solution. The matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

has complex eigenvalues $1 \pm i$. The corresponding eigenvectors are solutions of

$$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}, \qquad \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0},$$

and so we can take

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

The formula (5.5) holds still and we have

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Ae^{(1+i)t}\begin{pmatrix} 1 \\ i \end{pmatrix} + Be^{(1-i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}.$$

As $x(0) = y(0) = 1$ then

$$A + B = 1 \text{ and } i(A - B) = 1$$

so that $A = (1 - i)/2$ and $B = (1 + i)/2$. Hence

$$x(t) = \frac{(1-i)}{2}e^{(1+i)t} + \frac{(1+i)}{2}e^{(1-i)t} = \frac{e^t}{2}\left\{\left(e^{it} + e^{-it}\right) - i\left(e^{it} - e^{-it}\right)\right\} = e^t(\cos t + \sin t),$$

and similarly we find

$$y(t) = \frac{(i+1)}{2}e^{(1+i)t} + \frac{(-i+1)}{2}e^{(1-i)t} = \frac{e^t}{2}\left\{\left(e^{it} + e^{-it}\right) + i\left(e^{it} - e^{-it}\right)\right\} = e^t(\cos t - \sin t).$$

■

5.3 Epilogue — approximations*

To end this chapter here are three examples where, perhaps in the face of not being able to solve the differential equations exactly, it is still possible to solve similar differential equations which approximate the more complicated original problem. This material is not expressly on the syllabus; at the same time it would give a false impression of the study of ordinary differential equations to discuss only examples which could be solved exactly.

Example 75 *A mass P swinging on the end of a light rod is governed by the differential equation*

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{l}\sin\theta.$$

Suppose that the maximum swing of the mass is α. Show that the time T of an oscillation equals

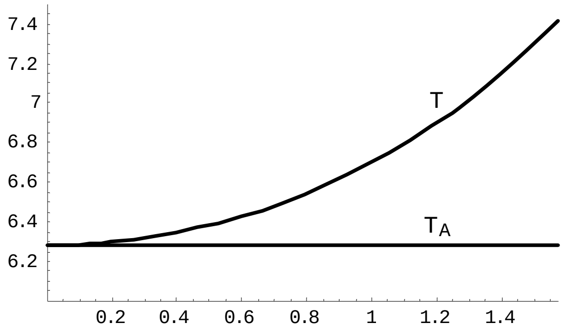
$$T = 4\sqrt{\frac{l}{2g}}\int_0^{\alpha}\frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\alpha}}$$

Solution. See Exercise 59. ■

Normally of course we make the assumption that the pendulum moves through small oscillations and approximate $\sin\theta \approx \theta$ to get the standard SHM equation, whose solutions have a period of

$$T_A = 2\pi\sqrt{\frac{l}{g}}.$$

Graphically we can see how the two compare



The approximation remains within a 1% error, that is $T/T_A < 1.01$, for $\theta < 0.398$ radians $\approx 22.8°$, and within a 5% error, that is $T/T_A < 1.05$, for $\theta < 0.874$ radians $\approx 50°$.

Definition 76 ***(Euler's Method)*** *This method suggests that an approximate solution to the initial-value problem*

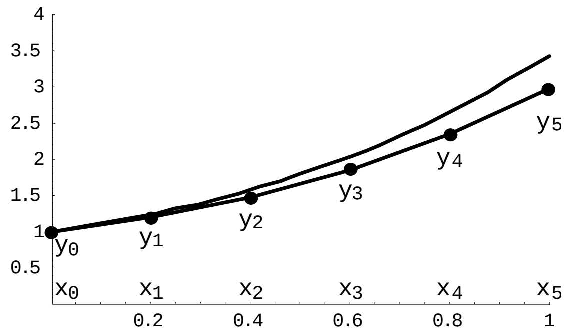
$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y), \qquad y(a) = y_0$$

can be found using the iteration

$$y_{n+1} = y_n + hf(a + nh, y_n)$$

where $y_k = y(a + kh)$ and h is the length of increment in x. The method works on the assumption that y will continue to grow on the range $nh < x < (n+1)h$ at roughly the same rate as it was growing at $x = a+nh, y = y_n$.

The smaller the h, the better the chance that this is a reasonable assumption.



Solving $y' = x + y$ with $y_0 = 1$ and $h = 0.2$

Example 77 *Find, using Euler's method, an approximation to $y(X)$ where y satisfies the initial-value problem*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y + x, \qquad y(0) = 1.$$

Use an increment of X/N and show that $y_N \to y(X)$ as $N \to \infty$. What value of N do we need to get $y(1)$ to within 5%? to within 1%?

Solution. Using integrating factors we can solve the equation exactly to get

$$y(x) = 2e^x - 1 - x.$$

The iteration for Euler's method is

$$y_{n+1} = y_n + h(nh + y_n) = (1+h)\,y_n + h^2 n, \qquad y(0) = 1,$$

where $h = X/N$. Now, either by induction or with knowledge of difference equations, we can solve this recursion to get

$$y_n = 2(1+h)^n - hn - 1.$$

Setting $h = X/N$ and $n = N$ we find

$$y_N = 2\left(1 + \frac{X}{N}\right)^N - X - 1.$$

It is a well-known limit, from analysis, that $(1 + X/N)^N \to e^X$ as $N \to \infty$ and so y_N does indeed tend to $y(X)$ as $N \to \infty$.

To answer the second part, we note with $X = 1$ that

$$y_N = 2\left(1 + \frac{1}{N}\right)^N - 2, \text{ whilst } y(1) = 2e - 2.$$

With a calculator or computer we can find that

$$\begin{aligned} \frac{y_N}{y(1)} &> 0.95 \text{ for } N \geqslant 15; \\ \frac{y_N}{y(1)} &> 0.99 \text{ for } N \geqslant 79. \end{aligned}$$

■

Example 78 ***(Critical Points and Stability)*** *The* ***Volterra-Lotka*** *equations are*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -mx + axy, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = by - kxy,$$

where a, b, k, m are positive constants, and they model competing numbers of predators x and prey y. The ***critical points*** *(x, y) are points where the populations x and y remain constant, i.e. where $\dot{x} = \dot{y} = 0$. Solving these equations we see that the two critical points are*

$$(0,0) \quad \text{and} \quad (b/k, m/a)\,.$$

What would happen though if the steady populations of $x = b/k, y = m/a$ were disturbed a little, e.g. some few extra predators or prey introduced?

Solution. To see this we make a change of variable $X = x - b/k$, $Y = y - m/a$. The Volterra-Lotka equations then become

$$\begin{aligned}
\frac{\mathrm{d}X}{\mathrm{d}t} &= -m\left(X + \frac{b}{k}\right) + a\left(X + \frac{b}{k}\right)\left(Y + \frac{m}{a}\right) = \frac{abY}{k} + aXY; \\
\frac{\mathrm{d}Y}{\mathrm{d}t} &= b\left(Y + \frac{m}{a}\right) - k\left(X + \frac{b}{k}\right)\left(Y + \frac{m}{a}\right) = -\frac{kmX}{a} - kXY.
\end{aligned}$$

But X and Y represent only the small changes in populations, and so the second order term XY will be smaller still and negligible for small enough pertubations. So we can approximate the equations to

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \left(\frac{ab}{k}\right)Y, \qquad \frac{\mathrm{d}Y}{\mathrm{d}t} = \left(\frac{-km}{a}\right)X.$$

The equations

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \left(\frac{ab}{k}\right)Y, \qquad \frac{\mathrm{d}Y}{\mathrm{d}t} = \left(\frac{-km}{a}\right)X$$

form a linear system of equations like the ones we studied in the previous section. We could solve them using similar methods, but that is really unnecessary here as we can note

$$\begin{aligned}
\frac{\mathrm{d}^2X}{\mathrm{d}t^2} &= \left(\frac{ab}{k}\right)\frac{\mathrm{d}Y}{\mathrm{d}t} = \left(\frac{ab}{k}\right)\left(\frac{-km}{a}\right)X = -bmX, \\
\frac{\mathrm{d}^2Y}{\mathrm{d}t^2} &= \left(\frac{-km}{a}\right)\frac{\mathrm{d}X}{\mathrm{d}t} = \left(\frac{-km}{a}\right)\left(\frac{ab}{k}\right)Y = -bmY,
\end{aligned}$$

which are SHM equations and X and Y are given by

$$X = \alpha_1 \cos\left(\sqrt{bm}t + \varepsilon_1\right), \qquad Y = \alpha_2 \cos\left(\sqrt{bm}t + \varepsilon_2\right), \tag{5.8}$$

for constants $\alpha_1, \alpha_2, \varepsilon_1, \varepsilon_2$. So the critical point $(b/k, m/a)$ is **stable**, meaning that if we perturb the populations from their steady states they remain roughly the same. In fact, from equations (5.8) we can see that the populations move around the critical point in a circular/elliptical motion. Unsurprisingly this type of critical point is called a **circular** one. ■

5.4 Exercises

Exercise 87 *Find the eigenvalues and two independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ of the matrix*

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}.$$

Put the vectors $\mathbf{v}_1, \mathbf{v}_2$ as the columns of a 2×2 matrix P. Find P^{-1} and verify by direct calculation that $P^{-1}AP$ is a diagonal matrix.

Exercise 88 ** Consider the following matrices, A, B, C, D. Where it makes sense to do so, calculate their sums and products.*

$$
\begin{aligned}
A &= \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 3 & 2 \\ 1 & 0 \end{pmatrix}, \\
C &= \begin{pmatrix} -1 & 2 & 3 \\ -2 & 1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} -1 & 12 \\ 6 & 0 \end{pmatrix}.
\end{aligned}
$$

Exercise 89 ** Let*

$$
A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ 0 & 3 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}. \tag{5.9}
$$

Calculate the products AB, BA, CA, BC.

Exercise 90 *Let*

$$
A = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}. \tag{5.10}
$$

Show that $AB = 0$, but that $BA \neq 0$.

Exercise 91 ** Let A and B be invertible $n \times n$ matrices. Show that $(AB)^{-1} = B^{-1}A^{-1}$. Give an example to show that $(A+B)^{-1} \neq A^{-1} + B^{-1}$ in general.*

Exercise 92 ** Determine whether the following matrices are invertible and find those inverses that do exist.*

$$
\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}, \qquad \begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & 5 & 4 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 3 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}.
$$

Exercise 93 ** Calculate A^{-1} when A is*

$$
\begin{pmatrix} 1 & -3 & 0 \\ 1 & -2 & 4 \\ 2 & -5 & 5 \end{pmatrix}.
$$

Exercise 94 *Find the general solution of the system*

$$
\frac{\mathrm{d}x}{\mathrm{d}t} = x + 4y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x + y.
$$

Find the solution $(x(t), y(t))$ to the above system for which $x(0) = 0$ and $y(0) = 2$. Sketch this curve in the xy-plane, and sketch in your eigenvectors $\mathbf{v}_1, \mathbf{v}_2$.

Exercise 95 *Find the solution of the system*

$$
\frac{\mathrm{d}x}{\mathrm{d}t} = -3x + \sqrt{2}y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{2}x - 2y,
$$

when $x(0) = \varepsilon_1, y(0) = \varepsilon_2$. Is the system's critical point $(0,0)$ stable?

Exercise 96 *Find the eigenvalues of, and two independent eigenvectors for, the matrix*

$$
\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}.
$$

(i) Find the general solution of the system

$$
\frac{\mathrm{d}x}{\mathrm{d}t} = 2x + 3y, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x + 4y.
$$

Solve

$$
\frac{\mathrm{d}x}{\mathrm{d}t} = 2x + 3y + 2e^{2t}, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = x + 4y + 3e^{2t}
$$

subject to the initial conditions $x(0) = \frac{-2}{3}$ and $y(0) = \frac{1}{3}$.

Exercise 97 *Find the general solution of the simultaneous equations*

$$\frac{\mathrm{d}x}{\mathrm{d}t}+2x-2y=t, \qquad \frac{\mathrm{d}y}{\mathrm{d}t}-3x+y=e^{t}.$$

Exercise 98 *A 2×2 matrix A is diagonalizable and has repeated eigenvalue λ. Show that $A=\lambda I_2$.*

Give an example of a 3×3 matrix A which is diagonalizable and which has a repeated eigenvalue but which isn't a multiple of I_3.

Exercise 99 ** Let*

$$A=\begin{pmatrix} 5 & -1 & -1 \\ 1 & 3 & 1 \\ -2 & 2 & 4 \end{pmatrix}.$$

Use formula (5.1) to find the eigenvalues of A, i.e. the roots of $\det(A-xI)$. Find an eigenvector corresponding to each eigenvalue.

Exercise 100 ** Find the general solution of the simultaneous differential equations*

$$\frac{\mathrm{d}x}{\mathrm{d}t}=5x-y-z, \qquad \frac{\mathrm{d}y}{\mathrm{d}t}=x+3y+z, \qquad \frac{\mathrm{d}z}{\mathrm{d}t}=-2x+2y+4z.$$

Exercise 101 ** Solve the following simultaneous differential equations for x,y,z as functions of t given that $x=-2, y=2, z=3$ when $t=0$ and*

$$x'(t)=x-2y+2z, \qquad y'(t)=x+y, \qquad z'(t)=2y-z.$$

Exercise 102 ** Show that 6 is a repeated eigenvalue of*

$$A=\begin{pmatrix} 6 & 1 & 2 \\ 0 & 7 & 2 \\ 0 & -2 & 2 \end{pmatrix},$$

and find two independent 6-eigenvectors. Hence find an invertible matrix P such that $P^{-1}AP$ is diagonal.

Exercise 103 ** Find the general solution of the simultaneous differential equations*

$$\frac{\mathrm{d}x}{\mathrm{d}t}=6x+y+2z, \qquad \frac{\mathrm{d}y}{\mathrm{d}t}=7y+2z, \qquad \frac{\mathrm{d}z}{\mathrm{d}t}=-2y+2z.$$

Exercise 104 ** Find a 2×2 matrix P with complex entries such that*

$$P^{-1}CP=\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

where C is as given in Example 62.

Exercise 105 ** Find the solution $y(x)$ of the initial-value problem*

$$\frac{\mathrm{d}y}{\mathrm{d}x}=x+2y, \qquad y(0)=0.$$

A numerical approximation to $y(x)$ is calculated using Euler's method:

$$x_n=nh, \qquad \frac{y_{n+1}-y_n}{h}=x_n+2y_n.$$

Show that there is a solution of the form

$$y_n=A(1+2h)^n+Bn^2+Cn+D$$

to the above recurrence relation which satisfies $y_0=0$, and confirm that, as $h\to 0$, the numerical approximation converges to the solution $y(x)$ of the initial-value problem.

Exercise 106 ** Show that $x_n = A\alpha^n + B\beta^n$ is the general solution of the recurrence relation*

$$x_{n+2} - (\alpha+\beta)\, x_{n+1} + \alpha\beta x_n = 0 \qquad (n \geqslant 0)$$

where $\alpha \neq \beta$. Find A and B in terms of y_0 and y_1.

Exercise 107 ** Solve the following recurrence relations*

$$\begin{array}{rrcll} (i) & x_{n+2} - 3x_{n+1} + 2x_n & = & 0; & \\ (ii) & x_{n+2} + 5x_{n+1} + 6x_n & = & 0 & x_0 = x_1 = 1; \\ (iii) & x_{n+2} + 2x_{n+1} + 2x_n & = & 0 & x_0 = x_1 = 1. \end{array}$$

Present your answer to part (iii) so that it is clear the solution is a real sequence.

Exercise 108 ** Solve the following recurrence relations*

$$\begin{array}{rrcll} (i) & x_{n+2} - 3x_{n+1} + 2x_n & = & n; & \\ (ii) & x_{n+2} + 5x_{n+1} + 6x_n & = & n^2 & x_0 = x_1 = 1. \end{array}$$

Exercise 109 ** (a) List the six bijections from the set $\{1, 2, 3\}$ to itself.*

(b) Every such bijection can be produced by repeatedly swapping just two elements at a time. Find two different ways of producing the bijection $1 \to 2$, $2 \to 3$, $3 \to 1$ as a composition of swaps.

(c) It is a theorem from the Hilary term algebra course that, for a given bijection, the number of swaps will always be even or always be odd; so these bijections are said to be even or odd. List the even bijections from $\{1, 2, 3\}$ to itself.

(d) Show that formula (5.1) agrees with the below formula

$$\sum_f \operatorname{sgn}(f)\, a_{1f(1)} a_{2f(2)} a_{3f(3)}$$

where the sum is taken over all bijections f from $\{1, 2, 3\}$ to itself and $\operatorname{sgn}(f) = 1$ or -1 depending on whether f is even or odd.

(e) The general $n \times n$ determinant formula is given by a similar expression summed over all bijections from $\{1, 2, \ldots, n\}$. How many such bijections are there?

Exercise 110 ** Let*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Find all those 2×2 matrices X which commute with A — i.e. those satisfying $XA = AX$.

Find all those 2×2 matrices Y which commute with B — i.e. those satisfying $YB = BY$.

Hence show that the only 2×2 matrices, which commute with all other 2×2 matrices, are scalar multiples of the identity matrix. [This result holds generally for $n \times n$ matrices.]

Exercise 111 ** Let A denote the 2×2 matrix*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Show that

$$A^2 - (\operatorname{trace} A)A + (\det A)I = 0 \tag{5.11}$$

where $\operatorname{trace} A = a + d$ is the trace of A, that is the sum of the diagonal elements, $\det A = ad - bc$ is the determinant of A, and I is the 2×2 identity matrix.

Suppose now that $A^n = 0$ for some $n \geqslant 2$. Prove that $\det A = 0$. Deduce using equation (5.11) that $A^2 = 0$.

Exercise 112 ** Let*

$$A = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}.$$

Show that

$$A^n = 3^{n-1} \begin{pmatrix} 2n+3 & -n \\ 4n & 3-2n \end{pmatrix}$$

for $n = 1, 2, 3, \ldots$. Can you find a matrix B such that $B^2 = A$?

6. PARTIAL DIFFERENTIATION

We began our study of ordinary differential equations (ODEs) by modelling the flight of a projectile. Let's consider here, before discussing **partial derivatives** and **partial differential equations** (PDEs), a slightly more complex example of a mug of tea cooling down on a table top.

In what sense is this new physical scenario more complicated to model? The quantity we are naturally interested in here is the temperature T of the liquid. In our earlier motivating example of the projectile, its height depended only on time t. Let's take the mug's dimensions as radius R and height H. Here the temperature of the tea will again depend on time t, but almost certainly not in a uniform way throughout the mug. So rather than just depending on time t the temperature will also depend on the spatial co-ordinates x, y and z. Also heat will be lost to the air, down the sides of the mug, to the air beyond and may heat up the table and surrounding air.

The differential equation governing the behaviour of $T(x, y, z, t)$ is a partial differential equation. The **heat equation**, (also more generally referred to as the **diffusion equation**), states that

$$\frac{\partial T}{\partial t} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

where κ is known as thermal diffusivity. These "fancy" derivatives with the curly ∂s just reflect that T depends on several variables.

- The partial derivative $\partial T/\partial t$ is the derivative of T with respect to time t when we keep all the other variables x, y, z constant.
- Again $\partial T/\partial t$ is a function of the four variables x, y, z, t.
- It is a measure of what we'd see if we focus separately on each point (x, y, z) and watch how the temperature changes over time. Because the tea is cooling we would expect that $\partial T/\partial t < 0$ throughout the mug and at all times.

What other information would we need to describe the tea's cooling? Well, we would again need an **initial condition** saying how hot the tea was to begin. So we might assume it to begin uniformly hot and have

$$T(x, y, z, 0) = T_0$$

for some initial temperature T_0 and for $x^2 + y^2 < R^2$, $0 < z < H$.

We would also need to make some assumptions about how the heat dissipated out of the mug. If we assume that air remains constantly at some ambient room temperature T_A then one **boundary condition** would be

$$T(x, y, H, t) = T_A$$

for $x^2 + y^2 < R^2$, $t > 0$; this describes the temperature's behaviour at the top of the mug (though it isn't a particularly realistic assumption physically!). If say the mug were insulated to allow no heat loss then another boundary condition would be

$$\frac{\partial T}{\partial z}(x, y, 0, t) = 0$$

for $x^2 + y^2 < R^2$, $t > 0$ at the base of the mug. Likewise down the sides of the mug we would have

$$\frac{\partial T}{\partial r}(x, y, z, t) = 0$$

for $x^2 + y^2 = R^2$, $0 < z < H$, $t > 0$ and where r denotes the distance of a point from the central axis of the mug.

Indeed, having noticed this symmetry in the cooling we might decide to change variables and consider this as a problem in just three variables r, t, z reducing the problem somewhat.

6.1 Partial Derivatives

Definition 79 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of n variables $x_1, x_2, \ldots, x_n$. Then the* ***partial derivative***

$$\frac{\partial f}{\partial x_i}(p_1, \ldots, p_n)$$

is the rate of change of f, at $(p_1, \ldots, p_n)$, when we vary only the variable x_i about p_i and keep all of the other variables constant. Precisely then

$$\frac{\partial f}{\partial x_i}(p_1, \ldots, p_n) = \lim_{h \to 0} \frac{f(p_1, \ldots, p_{i-1}, p_i + h, p_{i+1}, \ldots p_n) - f(p_1, \ldots, p_n)}{h}.$$

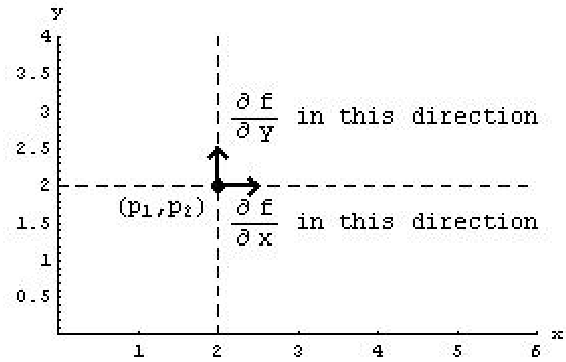
By contrast, derivatives such as df/dx are sometimes referred to as ***full derivatives****.*

- If $f(x)$ is a function of a single variable note that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial x}.$$

- $\partial f / \partial x$ is is still pronounced "d f by d x" or sometimes "partial d f by d x".
- We shall also, occasionally, write f_x for $\partial f / \partial x$, which is equally common notation.
- In some texts $\partial f / \partial x_i$ is denoted as f_i.

Notation 80



$$\frac{\partial f}{\partial x}(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}, \qquad \frac{\partial f}{\partial y}(a, b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k}.$$

- To stress which variables are being kept constant some texts use the notation

$$\left(\frac{\partial f}{\partial x}\right)_y$$

to denote that this is the partial derivative of f with respect to x whilst keeping y constant. This is a measure of how quickly $f(x, y)$ is changing as we move from the point (p_1, p_2) along the line $y = p_2$. We will however always may clear which **co-ordinate system** we are using and so a partial derivative will always denote differentiation with respect to some co-ordinate and all other co-ordinates in the system being kept constant.

Example 81 *Let*

$$f(x, y, z) = x^2 + ye^x + \frac{z}{y}.$$

Then

$$\frac{\partial f}{\partial x} = 2x + ye^x, \qquad \frac{\partial f}{\partial y} = e^x - \frac{z}{y^2}, \qquad \frac{\partial f}{\partial z} = \frac{1}{y}.$$

Example 82 *Let*

$$f(x_1,x_2,x_3,x_4)=\sin(x_1x_2)+\cos x_4.$$

Then

$$\frac{\partial f}{\partial x_1}=x_2\cos(x_1x_2), \qquad \frac{\partial f}{\partial x_2}=x_1\cos(x_1x_2), \qquad \frac{\partial f}{\partial x_3}=0, \qquad \frac{\partial f}{\partial x_4}=-\sin x_4.$$

Note that $\partial f/\partial x_3=0$ as the definition of f is independent of x_3.

- Note more generally that $\partial f/\partial x_3=0$ despite f being non-constant; f needs only to be constant as we vary x_3 and so the general solution of the equation $\partial f/\partial x_3=0$ is

$$f(x_1,x_2,x_3,x_4)=g(x_1,x_2,x_4)$$

for an arbitrary differentiable function g involving the other variables.

Example 83 *Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by*

$$f(x,y)=\begin{cases} x+y & \text{if } x=0 \text{ or } y=0,\\ 1 & \text{otherwise}\end{cases}$$

Note that the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at $(0,0)$ with

$$\begin{aligned}\frac{\partial f}{\partial x}(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{h-0}{h}=1,\\ \frac{\partial f}{\partial y}(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{h-0}{h}=1,\end{aligned}$$

yet f is not continuous at $(0,0)$. This contrasts with full derivatives where differentiability implies continuity (see Hilary term analysis course).

Definition 84 *We may define second and higher partial derivatives in a similar manner to how we define them for full derivatives. So, in the case of second partial derivatives of a function $f(x,y)$:*

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), && \text{also written } f_{xx}=(f_x)_x,\\ \frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), && \text{also written } f_{yx}=(f_y)_x,\\ \frac{\partial^2 f}{\partial y\partial x} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), && \text{also written } f_{xy}=(f_x)_y,\\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right), && \text{also written } f_{yy}=(f_y)_y.\end{aligned}$$

Example 85 *Let us return to the function*

$$f(x,y,z)=x^2+ye^x+\frac{z}{y}$$

from Example 81. Then

$$\frac{\partial f}{\partial x}=2x+ye^x, \qquad \frac{\partial f}{\partial y}=e^x-\frac{z}{y^2}, \qquad \frac{\partial f}{\partial z}=\frac{1}{y}.$$

So

$$\begin{aligned}&\frac{\partial^2 f}{\partial x^2}=2+ye^x, && \frac{\partial^2 f}{\partial y\partial x}=e^x, && \frac{\partial^2 f}{\partial z\partial x}=0,\\ &\frac{\partial^2 f}{\partial x\partial y}=e^x, && \frac{\partial^2 f}{\partial y^2}=\frac{2z}{y^3}, && \frac{\partial^2 f}{\partial z\partial y}=\frac{-1}{y^2},\\ &\frac{\partial^2 f}{\partial x\partial z}=0, && \frac{\partial^2 f}{\partial y\partial z}=\frac{-1}{y^2}, && \frac{\partial^2 f}{\partial z^2}=0.\end{aligned}$$

Note in the previous example

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}, \qquad \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial x \partial z}, \qquad \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z}.$$

This will typically be the case in the examples we'll see, as the following theorem shows:

Theorem 86 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ exist and are continuous. Then*

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Proof. (For reference only — requires Hilary term analysis, and non-examinable). For $x, y, h, k \in \mathbb{R}$ define

$$\phi(x, y) = f(x, y + k) - f(x, y), \quad \text{and} \quad \psi(x, y) = f(x + h, y) - f(x, y)$$

so that

$$\begin{aligned} D(x, y) &= f(x + h, y + k) - f(x + h, y) - f(x + h, y) + f(x, y) \\ &= \phi(x + h, y) - \phi(x, y) \\ &= \psi(x, y + k) - \psi(x, y). \end{aligned}$$

By the Mean-Value Theorem, applied twice, there exist $\theta_1, \theta_2 \in (0, 1)$ such that

$$\begin{aligned} D(x, y) &= \phi(x + h, y) - \phi(x, y) = h\phi_x(x + \theta_1 h, y) \\ &= h[f_x(x + \theta_1 h, y + k) - f_x(x + \theta_1 h, y)] \\ &= hk f_{xy}(x + \theta_1 h, y + \theta_2 k). \end{aligned}$$

Arguing similarly with from the $D(x, y) = \psi(x, y + k) - \psi(x, y)$ expression we know there exist $\theta_3, \theta_4 \in (0, 1)$ such that

$$D(x, y) = hk f_{yx}(x + \theta_3 h, y + \theta_4 k).$$

So

$$f_{xy}(x + \theta_1 h, y + \theta_2 k) = f_{yx}(x + \theta_3 h, y + \theta_4 k)$$

Letting h and k tend to 0, and using the continuity of f_{xy} and f_{yx} we see that $f_{xy} = f_{yx}$ as required. ■

Example 87 *Define $f : \mathbb{R}^2 \to \mathbb{R}$ by*

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that

$$\frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}$$

exist yet are unequal at $(0, 0)$.

Solution. We have for $x \neq 0$ and $y \neq 0$,

$$\frac{\partial f}{\partial x}(0, y) = \left[\frac{(x^2 + y^2) y (3x^2 - y^2) - xy (x^2 - y^2) 2x}{(x^2 + y^2)^2} \right] \Bigg|_{x=0} = \frac{-y^5}{y^4} = -y$$

and similarly

$$\frac{\partial f}{\partial y}(x, 0) = \left[\frac{(x^2 + y^2) x (x^2 - 3y^2) - xy (x^2 - y^2) 2y}{(x^2 + y^2)^2} \right] \Bigg|_{y=0} = \frac{x^5}{x^4} = x.$$

Hence

$$\frac{\partial^2 f}{\partial y \partial x}(0, 0) = -1 \neq 1 = \frac{\partial^2 f}{\partial x \partial y}(0, 0).$$

■

Example 88 *Find all solutions of the form $f(x,y)$ to the partial differential equations*

$$\text{(i)} \qquad \frac{\partial^2 f}{\partial y \partial x} = 0, \qquad \text{(ii)} \qquad \frac{\partial^2 f}{\partial x^2} = 0.$$

Solution. The first PDE is

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 0.$$

Those functions $g(x,y)$ which satisfy $\partial g/\partial y = 0$ are functions $p(x)$ which solely depend on x. So we have

$$\frac{\partial f}{\partial x} = p(x).$$

This looks like an equation we would normally just integrate up, not forgetting a constant. But again $\partial/\partial x$ sends to zero any function $Q(y)$ of y. So instead of a constant we have an arbitrary function of y. The solution then is

$$f(x,y) = P(x) + Q(y)$$

where $P(x)$ is an anti-derivative of $p(x)$, i.e. $P'(x) = p(x)$.

For the second equation $\partial^2 f/\partial x^2 = 0$ we can integrate in similar fashion to get $\partial f/\partial x = p(y)$ and then

$$f(x,y) = p(y)\,x + q(y)$$

where $p(y)$ and $q(y)$ are functions just of y. ■

Remark 89 Note how the solutions to these second order PDEs include two arbitrary **functions** rather than two arbitrary **constants** as is often the case of an ODE. This makes sense though when we note that partially differentiating wrt x annihilates functions solely in the variable y and not just constant functions.

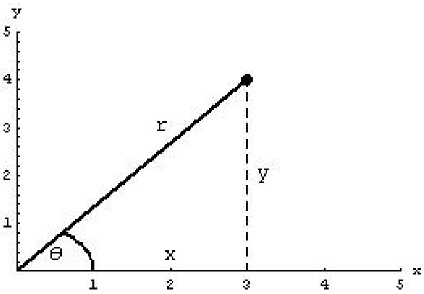
6.2 Co-ordinate Systems

In the examples we have seen so far we have considered functions $f(x,y)$ or $g(x,y,z)$ where we have been thinking of x, y, z as Cartesian co-ordinates. f and g are then functions defined on a 2-dimensional plane or in 3-dimensional space. There are other natural ways to place co-ordinates on a plane or space. Indeed, depending on the nature of a problem and any inherent symmetry, it may be very natural to use other co-ordinates.

In the next section we will derive the chain rule which describes the relationship between derivatives in one system and another. For now, we simply introduce some important examples of co-ordinate systems.

Example 90 ***(Planar Polar Co-ordinates)***

Instead of considering a point of a plane as so much along each of two perpendicular axes, we can equally describe a point P as being at a certain distance r from an origin O and such that OP makes an anti-clockwise angle θ with a fixed axis, usually the positive x-axis. This is shown in the diagram below.



We can see that the equations relating Cartesian co-ordinates and planar polar co-ordinates are

$$\begin{aligned} x &= r\cos\theta, \qquad y = r\sin\theta \\ r &= \sqrt{x^2+y^2}, \qquad \tan\theta = y/x. \end{aligned}$$

So in the above diagram where $x = 3$ and $y = 4$ then

$$r = \sqrt{3^2+4^2} = 5 \quad \textit{and} \quad \theta = \tan^{-1}\frac{4}{3} \approx 1.249 \textit{ radians.}$$

Note that r takes values in the range $r \in [0,\infty)$ and $\theta \in [0,2\pi)$,for example, or equally $(-\pi,\pi]$. Note also that θ is undefined at the origin.

Remark 91 Note the definition of partial derivative $\partial f/\partial x$ very much depends on the co-ordinate system that x is part of. It is important to know which other co-ordinates are being kept fixed. For example, we could have two different co-ordinate systems, one the standard Cartesian co-ordinates and the other being x and the polar co-ordinate θ. Consider now what $\partial r/\partial x$ means in each system. In Cartesian co-ordinates, we have

$$r = \sqrt{x^2+y^2} \text{ and so } \partial r/\partial x = x/\sqrt{x^2+y^2}.$$

However when we write r in terms of x and θ we have

$$r = x/\cos\theta \text{ and so } \frac{\partial r}{\partial x} = \frac{1}{\cos\theta} = \frac{\sqrt{x^2+y^2}}{x}$$

which is certainly a different answer! The reason is that the two derivatives we have calculated are

$$\left(\frac{\partial r}{\partial x}\right)_y \quad \text{and} \quad \left(\frac{\partial r}{\partial x}\right)_\theta$$

and so are measuring the change in x along curves $y = \text{const.}$ or along $\theta = \text{const.}$ which are very different directions. Indeed note that the two are equal only when $\cos^2\theta = 1$ in which case lines of constant θ and constant y are in the same direction.

Example 92 ***(Changing from polar to Cartesian co-ordinates and vice versa)***
Given a curve with equation $f(x,y) = 0$ then it can be rephrased as an equation in polar co-ordinates as

$$g(r,\theta) = f(r\cos\theta, r\sin\theta) = 0.$$

The unit circle $x^2+y^2 = 1$ clearly becomes $r = 1$ and the line $x = k$ becomes $r = k\sec\theta$.
More generally, the line $ax+by+c = 0$ becomes $r(a\cos\theta + b\sin\theta) = -c$ which can be rewritten as

$$r = A\sec(\theta-\alpha)$$

where $A = -c/\sqrt{a^2+b^2}$ and $\tan\alpha = b/a$.
In reverse, given a curve in polar co-ordinates $F(r,\theta) = 0$, then this can be rewritten as

$$G(x,y) = F\left(\sqrt{x^2+y^2}, \tan^{-1}\left(\frac{y}{x}\right)\right) = 0.$$

For example, $r = \cos\theta$ becomes $r^2 = r\cos\theta$ which gives $x^2+y^2 = x$. This we see is the circle

$$\left(x-\frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$$

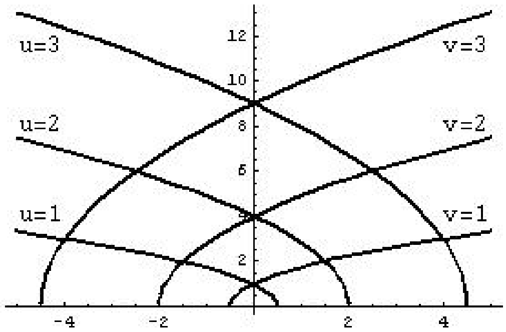
which is a circle of radius $1/2$ around the point $(1/2,0)$. Note the circle is mapped out as θ varies from $-\pi/2$ to $\pi/2$. As θ varies from $\pi/2$ to $3\pi/2$ then $\cos\theta$ is negative — when r is negative then (r,θ) is plotted in the opposite quadrant to $(-r,\theta)$ reflected through the origin — and so in this case as θ varies from $\pi/2$ to $3\pi/2$ the point (r,θ) retraces the circle once more.

Example 93 ***(Planar Parabolic Co-ordinates)***

Planar Parabolic co-ordinates u, v are given in terms of x and y by the relations

$$x = \frac{1}{2}\left(u^2 - v^2\right), \qquad y = uv.$$

Note that the curve $u = c$, in Cartesian co-ordinates, is $2xc^2 = c^4 - y^2$ and the curve $v = k$, in Cartesian co-ordinates is $2xk^2 = y^2 - k^4$ both of which are parabolas. As u and v vary over the positive numbers then (x, y) varies over the upper half-plane.



Example 94 ***(Cylindrical Polar Co-ordinates)***

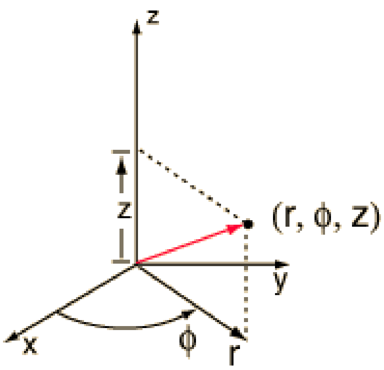
We can naturally extend planar polar co-ordinates into three dimensions using a further z co-ordinate. In this case they are called cylindrical polar co-ordinates. The relationships between r, ϕ, z and x, y, z are given by

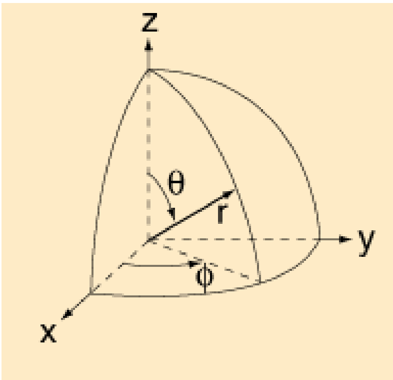
$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = z$$

and

$$r = \sqrt{x^2 + y^2}, \qquad \tan\phi = \frac{y}{x}, \qquad z = z.$$

Note that $r =$ const. defines a cylinder, $\phi =$ const. defines a vertical plane through the origin and $z =$ const. defines a horizontal plane.





Cylindrical Polars *Spherical Polars*

Example 95 ***(Spherical Polar Co-ordinates)***

In the same way that latitude and longitude are used to determine a position on the planet, we can similarly use two angles θ and ϕ to determine position on concentric spheres distance r from the origin. These are called spherical polar co-ordinates.

Here r is simply the distance of a point P from the origin O. The angle θ is the angle OP makes with the vertical z-axis and takes values in the range $-\pi/2 \leqslant \theta \leqslant \pi/2$. Finally, if Q is the projection of P vertically into the xy-plane then ϕ is the angle OQ makes with the positive x-axis. The relationships between x, y, z and r, ϕ, θ are given by

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

and

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \tan\phi = \frac{y}{x}, \qquad \tan\theta = \frac{\sqrt{x^2 + y^2}}{z}.$$

6.3 Exercises

Exercise 113 *Let* $f(x,y) = \exp(y/x)$ *where* $x \neq 0$. *Find the partial derivatives* f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy}, *verifying that* $f_{xy} = f_{yx}$.

Exercise 114 *If* $f(x,y,z) = \left(x^2+y^2+z^2\right)^{-1/2}$ *for* $(x,y,z) \neq (0,0,0)$, *then verify that* $f_{xx} + f_{yy} + f_{zz} = 0$.

Exercise 115 *Let* $F(x,y) = f(x/y)$ *where* f *is a differentiable function of one variable. Show that*

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = 0.$$

Exercise 116 *Let* $F(x,y) = f(y\ln x)$ *where* f *is a differentiable function of one variable. Show that*

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = xy\frac{\partial^2 F}{\partial x \partial y} - x^2 \ln x \frac{\partial^2 F}{\partial x^2}.$$

Exercise 117 *Let* $F(x,t) = f(x-ct) + g(x+ct)$ *where* f *and* g *are differentiable functions of one variable and* c *is a constant. Show that* $c^2 F_{xx} = F_{tt}$.

Exercise 118 *If* $x^2/a^2 + y^2/b^2 = 1$ *find* $\mathrm{d}y/\mathrm{d}x$ *and* $\mathrm{d}^2y/\mathrm{d}x^2$ *by implicit differentiation.*

Exercise 119 *Find the point on the curve* $x^2 + xy + y^2 = 1$ *which is closest to the line* $x + y = 2$.

Exercise 120 *By changing to polar co-ordinates, or otherwise, sketch the curve* $\left(x^2+y^2\right)^3 = x^4$.

Exercise 121 *Sketch the logarithmic spiral* $r = ae^{b\theta}$ *where* a, b *are positive constants.*
Show that the angles between the tangent to the curve and the radius from $(0,0)$ *to* (r,θ), *is always constant.*
Show further that the curve's arc-length from the origin (as $\theta \to -\infty$*) to the point* $(r,0)$ *is finite and determine this arc length in terms of* a *and* b.

Exercise 122 *Polar co-ordinates* r *and* θ *are defined for* $x > 0$, $y \in \mathbb{R}$ *by* $r = \sqrt{x^2+y^2}$, *and* $\theta = \tan^{-1}(y/x)$.
(i) Sketch the curves $r =$ *const. and* $\theta =$ *const. At what points is* $\partial r/\partial y$ *positive? At what points is* $\partial y/\partial r$ *positive?*
(ii) Find $\frac{\partial r}{\partial x}$, $\frac{\partial \theta}{\partial x}$, $\frac{\partial r}{\partial y}$, $\frac{\partial \theta}{\partial y}$, $\frac{\partial x}{\partial r}$, $\frac{\partial y}{\partial r}$, $\frac{\partial x}{\partial \theta}$, $\frac{\partial y}{\partial \theta}$. *Verify that* $\frac{\partial r}{\partial y}\frac{\partial y}{\partial r} < 1$ *at all points.*

Exercise 123 *The variables* u *and* v *are defined in terms of* $x \in \mathbb{R}$ *and* $y > 0$ *by*

$$u = \sqrt{x^2+y^2} + x, \qquad v = \sqrt{x^2+y^2} - x.$$

(i) Sketch the following curves $u = i$ *and* $v = i$ *for* $i = 1,2,3$. *[N.B. remember that* $y > 0$.*] From your diagram only, explain why* $\partial x/\partial u > 0$ *and* $\partial y/\partial u > 0$ *at every point.*
(ii) Write x *and* y *as functions of* u *and* v. *Find* $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial y}$, $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial u}$, $\frac{\partial y}{\partial v}$.

Exercise 124 *Show that the function* $f(x,y) = (Ax^n + Bx^{-n})(C\cos ny + D\sin ny)$, *where* A, B, C, D, n *are real constants and* n *is non-zero, satisfies the partial differential equation*

$$\frac{\partial^2 f}{\partial x^2} + \frac{1}{x}\frac{\partial f}{\partial x} + \frac{1}{x^2}\frac{\partial^2 f}{\partial y^2} = 0.$$

Exercise 125 *What functions* $f(x,y,z)$ *satisfy the following partial differential equations?*

$$\text{(i)} \quad \frac{\partial^3 f}{\partial z^3} = 0; \qquad \text{(ii)} \quad \frac{\partial^3 f}{\partial x \partial y \partial z} = 0.$$

Exercise 126 *What functions* $f(x,y)$ *satisfy the following partial differential equations?*

$$\text{(i)} \quad \frac{\partial f}{\partial x} + f = \sin y; \qquad \text{(ii)} \quad \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x} = 0.$$

Exercise 127 *What functions $f(x,y)$ satisfy the following partial differential equations?*

$$(i)\quad \frac{\partial^2 f}{\partial x^2} - f = 0; \qquad (ii)\quad \frac{\partial^2 f}{\partial x^2} - f = e^{xy}.$$

What value does your solution to the second equation take on the lines $y = \pm 1$?

Exercise 128 *Let $x \in \mathbb{R}$ and $t > 0$. Verify that*

$$T(x,t) = \frac{A}{\sqrt{t}} \exp\left(\frac{-x^2}{4\kappa t}\right)$$

is a solution of the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}. \tag{6.1}$$

Sketch T as a function of x at two different times t. Show that

$$\int_{-\infty}^{\infty} T(x,t)\,\mathrm{d}x$$

is a constant function of t.

Exercise 129 *Show that*

$$T(x,t) = \int_0^{x/\left(2\sqrt{\kappa t}\right)} e^{-s^2}\,\mathrm{d}s$$

satisfies the heat equation (6.1).

Exercise 130 *Verify, for $y > 0$, that*

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\int_0^1 \sin(xy)\,\mathrm{d}x\right) = \int_0^1 \frac{\partial}{\partial y}(\sin xy)\,\mathrm{d}x,$$

Exercise 131 ** Let*

$$I(y) = \int_0^{\infty} e^{-xy}\frac{\sin x}{x}\,\mathrm{d}x$$

where $y > 0$. Given that

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\int_0^{\infty} e^{-xy}\frac{\sin x}{x}\,\mathrm{d}x\right) = \int_0^{\infty} \frac{\partial}{\partial y}\left(e^{-xy}\frac{\sin x}{x}\right)\mathrm{d}x,$$

calculate $I'(y)$. What is the limit $\lim_{y\to\infty} I(y)$? Deduce that

$$I(y) = \frac{\pi}{2} - \tan^{-1} y.$$

Exercise 132 ** Given that*

$$\frac{\mathrm{d}}{\mathrm{d}a}\left(\int_0^{\infty} e^{-x^2}\cos ax\,\mathrm{d}x\right) = \int_0^{\infty} \frac{\partial}{\partial a}\left(e^{-x^2}\cos ax\right)\mathrm{d}x,$$

show that

$$\int_0^{\infty} e^{-x^2}\cos ax\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}\exp\left(\frac{-a^2}{4}\right).$$

Exercise 133 ** The Legendre polynomials $P_n(x)$ (see Exercises 85 and 86) can be defined by means of a generating series; to be precise we can define them by the equation*

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)\,t^n, \qquad |t| < 1.$$

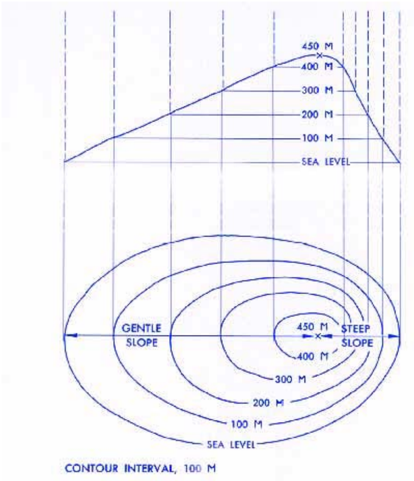
By differentiating partially with respect to t, or otherwise, show that

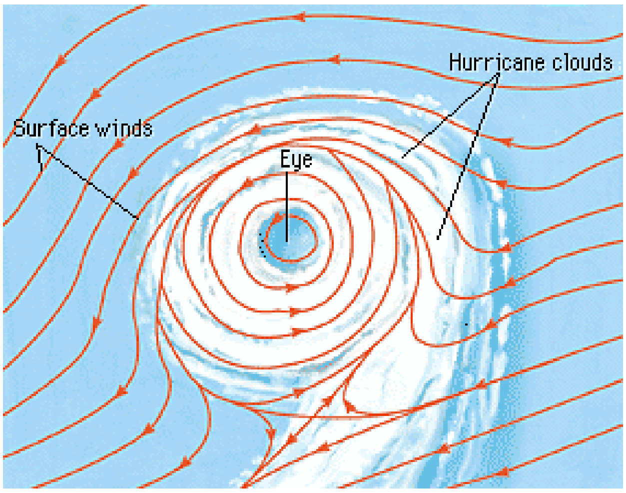
$$(n+1)\,P_n(x) = (2n+1)\,xP_n(x) - nP_{n-1}(x).$$

7. CHANGE OF VARIABLE. CHAIN RULE

7.1 Functions of Two Variables

Functions of more than one variable are common throughout mathematics. The motivating example for partial derivatives at the start of the last chapter involved a temperature T which depended on three spatial co-ordinates x, y, z and one temporal co-ordinate t. Such functions are often associated with maps as well. For example, we might have a physical map denoting the height z above a point (x, y). The map might also include contours which are the curves $z = c$ of constant height. Here z is a **scalar** function of two variables.





Or the map might be a meteorological map denoting the wind speed and direction (at a fixed height) above a point (x, y). Above is a wind-direction field associated with a hurricane — with each point (x, y) is associated a *vector* $\mathbf{u}(x, y)$. That is, $\mathbf{u}$ is a **vector-valued** function of two variables.

Example 96 *The height z of a col above the point (x, y) is given by the function*

$$f(x, y) = 10 - (x-1)^2 + y^2.$$

Write this as a function $g(r, \theta)$ of planar polar co-ordinates.

Solution. As $x = r\cos\theta$ and $y = r\sin\theta$ then we can write

$$\begin{aligned} f(x, y) &= 10 - (r\cos\theta - 1)^2 + (r\sin\theta)^2 \\ &= 10 - r^2\cos^2\theta + 2r\cos\theta - 1 + r^2\sin^2\theta \\ &= 9 - r^2\cos 2\theta + 2r\cos\theta \\ &= g(r, \theta). \end{aligned}$$

■

Remark 97 Note that f and g are different functions, even though $z = f(x, y) = g(r, \theta)$. It is **not** the case that $z = f(r, \theta)$, rather z is given by a different rule in terms of r and θ. The rule

$$z = f(r, \theta) = 10 - (r-1)^2 + \theta^2$$

is clearly not the right one!

7.2 The Chain Rule

The *chain rule for two or more variables* serves the same purpose as the chain rule for one variable. Recall that the one-variable chain rule states that

$$\frac{\mathrm{d}f}{\mathrm{d}x}=\frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

The rule arises when we wish to calculate the derivative of the composition of two functions $f\left(u\left(x\right)\right)$ with respect to x.

Likewise we might have a function $f\left(u,v\right)$ of two variables u and v, each of which are functions of variables x and y. We can then make the composition F

$$F\left(x,y\right)=f\left(u\left(x,y\right),v\left(x,y\right)\right),$$

which is itself a function of x and y. We might then wish to calculate its partial derivatives

$$\frac{\partial F}{\partial x}\text{ and }\frac{\partial F}{\partial y}.$$

The chain rule states that

$$\begin{aligned}\frac{\partial F}{\partial x}&=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x},\\ \frac{\partial F}{\partial y}&=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}.\end{aligned}$$

Before going on to prove the chain rule, here is an example approached two different ways.

Example 98 *Let*

$$f\left(u,v\right)=\left(u-v\right)\sin u+e^{v},\qquad u\left(x,y\right)=x^{2}+y,\qquad v\left(x\right)=y-2x,$$

and let $F\left(x,y\right)=f\left(u\left(x,y\right),v\left(x,y\right)\right)$. Calculate $\partial F/\partial x$ and $\partial F/\partial y$ by (i) direct calculation, (ii) the chain rule.

Solution. (i) We have that

$$F\left(x,y\right)=\left(x^{2}+2x\right)\sin\left(x^{2}+y\right)+\exp\left(y-2x\right).$$

Hence

$$\begin{aligned}\frac{\partial F}{\partial x}&=\left(2x+2\right)\sin\left(x^{2}+y\right)+2x\left(x^{2}+2x\right)\cos\left(x^{2}+y\right)-2\exp\left(y-2x\right);\\ \frac{\partial F}{\partial y}&=\left(x^{2}+2x\right)\cos\left(x^{2}+y\right)+\exp\left(y-2x\right).\end{aligned}$$

(ii) Using the chain rule we have

$$\begin{aligned}\frac{\partial F}{\partial x}&=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\\ &=\left(\sin u+\left(u-v\right)\cos u\right)2x+\left(-\sin u+e^{v}\right)\left(-2\right)\\ &=\left(2x+2\right)\sin\left(x^{2}+y\right)+2x\left(x^{2}+2x\right)\cos\left(x^{2}+y\right)-2\exp\left(y-2x\right);\\ \frac{\partial F}{\partial y}&=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\\ &=\left(\sin u+\left(u-v\right)\cos u\right)\left(1\right)+\left(-\sin u+e^{v}\right)\left(1\right)\\ &=\left(x^{2}+2x\right)\cos\left(x^{2}+y\right)+\exp\left(y-2x\right).\end{aligned}$$

■

Theorem 99 ***(Chain Rule)*** *Let $F(t) = f(u(t), v(t))$ with u and v differentiable and f being continuously differentiable in each variable. Then*

$$\frac{dF}{dt} = \frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt}$$

Proof. (Not examinable) If we change t to $t + \delta t$, let δu and δv be the corresponding changes in u and v. Then

$$\delta u = \left(\frac{\mathrm{d}u}{\mathrm{d}t} + \varepsilon_1\right)\delta t, \quad \text{and} \quad \delta v = \left(\frac{\mathrm{d}v}{\mathrm{d}t} + \varepsilon_2\right)\delta t,$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $\delta t \to 0$. Now

$$\begin{aligned} \delta F &= f(u + \delta u, v + \delta v) - f(u, v) \\ &= [f(u + \delta u, v + \delta v) - f(u, v + \delta v)] + [f(u, v + \delta v) - f(u, v)] \end{aligned}$$

By the Mean-value Theorem (Hilary term Analysis) we have

$$\begin{aligned} f(u + \delta u, v + \delta v) - f(u, v + \delta v) &= \delta u \frac{\partial f}{\partial u}(u + \theta_1 \delta u, v + \delta v), \\ f(u, v + \delta v) - f(u, v) &= \delta v \frac{\partial f}{\partial v}(u, v + \theta_2 \delta v), \end{aligned}$$

for some $\theta_1, \theta_2 \in (0, 1)$. By the continuity of f_u and f_v then we have

$$\begin{aligned} \delta u \frac{\partial f}{\partial u}(u + \theta_1 \delta u, v + \delta v) &= \delta u \left(\frac{\partial f}{\partial u}(u, v) + \eta_1\right) \\ \delta v \frac{\partial f}{\partial v}(u, v + \theta_2 \delta v) &= \delta v \left(\frac{\partial f}{\partial v}(u, v) + \eta_2\right) \end{aligned}$$

where $\eta_1, \eta_2 \to 0$ as $\delta u, \delta v \to 0$. So, putting this all together

$$\begin{aligned} \frac{\delta F}{\delta t} &= \frac{\delta u}{\delta t}\left(\frac{\partial f}{\partial u}(u, v) + \eta_1\right) + \frac{\delta v}{\delta t}\left(\frac{\partial f}{\partial v}(u, v) + \eta_2\right) \\ &= \left(\frac{\mathrm{d}u}{\mathrm{d}t} + \varepsilon_1\right)\left(\frac{\partial f}{\partial u}(u, v) + \eta_1\right) + \left(\frac{\mathrm{d}v}{\mathrm{d}t} + \varepsilon_2\right)\left(\frac{\partial f}{\partial v}(u, v) + \eta_2\right). \end{aligned}$$

Letting $\delta t \to 0$ we get the required result. ■

Corollary 100 *Let $F(x, y) = f(u(x, y), v(x, y))$ with u, v differentiable in each variable and f being continuously differentiable in each. Then*

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}, \qquad \frac{\partial F}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}$$

Proof. This follows easily from the previous theorem by treating first x and then y as constants when differentiating. ■

Example 101 *A particle $P(x, y, z)$ moves in three dimensional space on a helix so that at time t*

$$x(t) = \cos t, \qquad y(t) = \sin t, \qquad z(t) = t.$$

The temperature T at (x, y, z) equals $xy + yz + zx$. Use (i) direct calculation, (ii) the chain rule, to calculate dT/dt.

Solution. (i)

$$T(t) = x(t)\,y(t) + y(t)\,z(t) + z(t)\,x(t) = \cos t \sin t + t \sin t + t \cos t.$$

So

$$\frac{dT}{dt} = -\sin^2 t + \cos^2 t + \sin t + \cos t + t \cos t - t \sin t.$$

(ii) Alternatively the chain rule says that

$$\begin{aligned}
\frac{dT}{dt} &= \frac{\partial T}{\partial x}\frac{dx}{dt}+\frac{\partial T}{\partial y}\frac{dy}{dt}+\frac{\partial T}{\partial z}\frac{dz}{dt} \\
&= (y+z)(-\sin t)+(x+z)\cos t+(x+y) \\
&= (t+\sin t)(-\sin t)+(t+\cos t)\cos t+(\sin t+\cos t) \\
&= -\sin^2 t+\cos^2 t+\sin t+\cos t+t\cos t-t\sin t.
\end{aligned}$$

■

Example 102 *Let $z=f(xy)$ where f is an arbitrary differentiable function in one variable. Show that*

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=0. \tag{7.1}$$

Solution. By the chain rule

$$\frac{\partial z}{\partial x}=yf'(xy) \quad \text{and} \quad \frac{\partial z}{\partial y}=xf'(xy).$$

Hence

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=xyf'(xy)-yxf'(xy)=0.$$

■

Example 103 *Conversely show that any solution of (7.1) is of the form $z=f(xy)$.*

Solution. We first make a change of co-ordinates

$$u=y/x, \qquad v=xy.$$

We are aiming to show that z is a function solely in v, or equivalently that $\partial z/\partial u=0$. By the chain rule

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}.$$

Solving for x and y in terms of u and v we have that

$$x=\sqrt{\frac{v}{u}} \quad \text{and} \quad y=\sqrt{uv}.$$

Then

$$\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{-1}{2}\sqrt{\frac{v}{u^3}}\frac{\partial z}{\partial x}+\frac{1}{2}\sqrt{\frac{v}{u}}\frac{\partial z}{\partial y} \\
&= \frac{-1}{2}\frac{x^2}{y}\frac{\partial z}{\partial x}+\frac{1}{2}x\frac{\partial z}{\partial y} \\
&= \frac{-1}{2}\frac{x}{y}\left(x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}\right)=0
\end{aligned}$$

and $z=f(v)=f(xy)$ as required, where f is an arbitrary differentiable function. ■

Example 104 *A particle P moves around on the unit sphere $r=1$. Find P's velocity $\mathbf{v}(t)=d\mathbf{r}/dt$ in terms of the spherical polar co-ordinates ϕ,θ, and their derivatives and verify by direct calculation that $\mathbf{v}\cdot\mathbf{r}=0$.*

Solution. Recall that

$$x=\sin\theta\cos\phi, \qquad y=\sin\theta\sin\phi, \qquad z=\cos\theta,$$

where θ and ϕ are functions of t. Now

$$\begin{aligned}
\dot{x} &= \cos\theta\cos\phi\,\dot{\theta}-\sin\theta\sin\phi\,\dot{\phi}, \\
\dot{y} &= \cos\theta\sin\phi\,\dot{\theta}+\sin\theta\cos\phi\,\dot{\phi}, \\
\dot{z} &= -\sin\theta\,\dot{\theta}.
\end{aligned}$$

So

$$
\begin{aligned}
\mathbf{v}\cdot\mathbf{r} &= \left(\cos\theta\cos\phi\,\dot{\theta}-\sin\theta\sin\phi\,\dot{\phi}\right)\sin\theta\cos\phi \\
&\quad+\left(\cos\theta\sin\phi\,\dot{\theta}+\sin\theta\cos\phi\,\dot{\phi}\right)\sin\theta\sin\phi \\
&\quad+\left(-\sin\theta\,\dot{\theta}.\right)\cos\theta \\
&= \dot{\theta}\left(\cos\theta\cos^2\phi\sin\theta+\cos\theta\sin^2\phi\sin\theta-\cos\theta\sin\theta\right) \\
&\quad+\dot{\phi}\left(-\sin^2\theta\sin\phi\cos\phi+\sin^2\theta\sin\phi\cos\phi\right) \\
&= 0
\end{aligned}
$$

■

This is true for any movement on the sphere. We can prove this much more easily by differentiating the vector identity $\mathbf{r}\cdot\mathbf{r}=1$ to get $2\mathbf{v}\cdot\mathbf{r}=0$.

To find the particle's acceleration by means of a chain rule we would need the next theorem.

Theorem 105 *(The Second Order Chain Rule)*
Let $F(x,y)=f(u(x,y),v(x,y))$. Then

$$
\begin{aligned}
F_{xx} &= f_u u_{xx}+f_v v_{xx}+f_{uu}(u_x)^2+2f_{uv}v_x u_x+f_{vv}(v_x)^2, \\
F_{xy} &= f_u u_{xy}+f_v v_{xy}+f_{uu}u_x u_y+f_{uv}(v_y u_x+v_x u_y)+f_{vv}v_x v_y, \\
F_{yy} &= f_u u_{yy}+f_v v_{yy}+f_{uu}(u_y)^2+2f_{uv}v_y u_y+f_{vv}(v_y)^2.
\end{aligned}
$$

Proof.

$$
\begin{aligned}
F_{xx} &= (f_u u_x+f_v v_x)_x \\
&= (f_u)_x u_x+(f_v)_x v_x+f_u u_{xx}+f_v v_{xx} \\
&= (f_{uu}u_x+f_{uv}v_x)u_x+(f_{vu}u_x+f_{vv}v_x)v_x+f_u u_{xx}+f_v v_{xx} \\
&= f_u u_{xx}+f_v v_{xx}+f_{uu}(u_x)^2+2f_{uv}v_x u_x+f_{vv}(v_x)^2
\end{aligned}
$$

and the other results follow similarly. ■

Example 106 *Let $u=x^2-y^2$ and $v=x^2-y$ for all $x,y\in\mathbb{R}$, and let $f(x,y)=x^2+y^2$.*
(a) Find the partial derivatives u_x,u_y,x_u,y_u in terms of x and y. For which values of x and y are your results valid?
(b) Find f_u and f_v. Hence show that $f_u+f_v=1$, and that $f_{uu}=f_{vv}$. Evaluate f_{uu} in terms of x and y.

Solution. (a) Clearly $u_x=2x$ and $u_y=-2y$. We can also differentiate the expression for u and v with respect to u to get

$$1=2xx_u-2yy_u,\qquad 0=2xx_u-y_u.$$

Solving the simultaneous equations in x_u and y_u gives

$$x_u=\frac{1}{2x(1-2y)},\qquad y_u=\frac{1}{1-2y}.$$

These expressions are clearly valid when $x\neq 0, y\neq 1/2$.
(b) By the chain rule

$$f_u=f_x x_u+f_y y_u=\frac{2x}{2x(1-2y)}+\frac{2y}{1-2y}=\frac{1+2y}{1-2y}.$$

If we calculate x_v and y_v in a similar fashion to the above calculations for x_u and y_u we obtain

$$x_v=\frac{y}{x(2y-1)},\qquad y_v=\frac{1}{2y-1}.$$

So, by the chain rule again,

$$f_v = f_x x_v + f_y y_v = \frac{2xy}{x(2y-1)} + \frac{2y}{2y-1} = \frac{4y}{2y-1}.$$

Hence

$$f_u + f_v = \frac{(1+2y)+(-4y)}{1-2y} = 1.$$

Now differentiating the identity $f_u + f_v = 1$ separately with respect to u and v gives

$$f_{uu} + f_{vu} = 0 = f_{uv} + f_{vv}.$$

As $f_{uv} = f_{vu}$ then $f_{uu} = f_{vv}$ as required. To find f_{uu} explicitly we can use the chain rule again to note

$$f_{uu} = (f_u)_x x_u + (f_u)_y y_u = 0 x_u + \frac{\partial}{\partial y}\left(\frac{1+2y}{1-2y}\right)\frac{1}{1-2y} = \frac{4}{(1-2y)^3}.$$

■

7.3 Exercises

Exercise 134 *Suppose* $f(x,y) = x^2 + y^2$ *where*

$$x(t) = t\cos t \quad \text{and} \quad y(t) = t\sin t.$$

Let $F(t) = f(x(t), y(t))$. *Find* $\mathrm{d}F/\mathrm{d}t$ *in two ways: (i) directly, (ii) using the chain rule.*

Exercise 135 *Use the chain rule to find* $\partial F/\partial u$ *and* $\partial F/\partial v$ *when* $F(u,v) = f(x,y)$ *and*

$$f(x,y) = y/x, \qquad x = u+v, \qquad y = u-v.$$

Exercise 136 *Let* $z(x,y) = \frac{1}{x}g(y/x)$, *where* g *is a differentiable function of one variable. Show that*

$$xz_x + yz_y + z = 0.$$

Show conversely that the only solutions of this partial differential equation are of the form $z(x,y) = g(x,y)/x$.

Exercise 137 *Let* $f(x,y) = xg(y/x)$ *where* g *is a differentiable function in one variable. Show that*

$$x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = 0.$$

Exercise 138 *Let* $u = x+y$ *and* $v = xy$, *and let* f *be a suitably differentiable function in* x *and* y. *Express* f_x *and* f_y *in terms of* f_u *and* f_v. *Show further that*

$$f_{xy} = f_{uu} + u f_{uv} + v f_{vv} + f_v.$$

Exercise 139 *Suppose that* f *is a function of* x *and* y, *and that* y *is a function of* x. *Show that*

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x, y(x)) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$

If $y(x)$ *is defined implicitly by the equation* $f(x,y) = 0$, *deduce that*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial f}{\partial x}\bigg/\frac{\partial f}{\partial y}.$$

Show further that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{(f_y)^3}\left\{2 f_x f_y f_{xy} - (f_x)^2 f_{yy} - (f_y)^2 f_{xx}\right\}$$

Exercise 140 *The equation* $3y = z^3 + 3xz$ *defines* z *implicitly in terms of* x *and* y. *Evaluate* z_{xx}, z_{xy} *and* z_{yy} *in terms of* x *and* y. *Verify that* z *is a solution of*

$$z_{xx} + xz_{yy} = 0.$$

Exercise 141 *Find* $\partial z/\partial s$ *and* $\partial z/\partial t$ *given that*

$$\begin{aligned} z &= x^2 + xy, \\ x^2 + y^3 &= st + 5, \\ x^3 - y^2 &= s^2 + t^2. \end{aligned}$$

Exercise 142 *Given* $xs^2 + yt^2 = 1$ *and* $x^2 s + y^2 t = xy - 4$ *find* $\partial x/\partial s$, $\partial x/\partial t$, $\partial y/\partial s$, $\partial y/\partial t$ *at*

$$(x, y, s, t) = (1, -3, 2, -1).$$

Exercise 143 *Let* $w = x^2 + xy + z^2$ *and*

$$t = \frac{x^3 + x}{3} = \frac{y^4 + y}{4} = \frac{z^5 + z}{5}.$$

Find $\mathrm{d}w/\mathrm{d}t$.

Exercise 144 *Let* $x = e^u \cos v$ *and* $y = e^u \sin v$ *and let* $f(x, y) = g(u, v)$. *Show that*

$$\left(x^2 + y^2\right)(f_{xx} + f_{yy}) = g_{uu} + g_{vv}.$$

Exercise 145 *Variables* u *and* v *are given in terms of* x *and* y *by*

$$u = x^2 - y^2, \qquad v = 2xy,$$

and let $g(u, v) = f(x, y)$. *Using the chain rule show that*

$$\frac{\partial^2 f}{\partial x^2} = 2\frac{\partial g}{\partial u} + 4x^2\frac{\partial^2 g}{\partial u^2} + 8xy\frac{\partial^2 g}{\partial u \partial v} + 4y^2\frac{\partial^2 g}{\partial v^2}$$

and find a similar expression for $\partial^2 f/\partial y^2$. *Hence express* $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2$ *in terms of the partial derivatives of* g. *Deduce that if* $f(x, y) = x + y$ *then*

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 0.$$

Exercise 146 *The function* $f(x, y)$ *satisfies the equation*

$$y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0.$$

Let $F(u, v) = f(x, y)$ *where*

$$u = x^2 - y^2 \quad \text{and} \quad v = 2xy.$$

Use the chain rule to find $\partial f/\partial x$ *and* $\partial f/\partial y$ *in terms of* $\partial F/\partial u$ *and* $\partial F/\partial v$. *Hence show that* $\partial F/\partial v = 0$ *and that* f *is, in fact, a function of* $x^2 - y^2$ *only.*

Exercise 147 *Let* $z(x, y) = f(xy)$ *where* f *is a twice-differentiable function of one variable. Show that* z *satisfies the equation*

$$x^2 z_{xx} + y^2 z_{yy} + xz_x + yz_y = 2xyz_{xy}.$$

Exercise 148 *If* $f : \mathbb{R}^n \to \mathbb{R}$ *has continuous partial derivatives and is homogeneous of degree* k, *that is*

$$f(t\mathbf{x}) = t^k f(\mathbf{x})$$

for any $\mathbf{x} \in \mathbb{R}^n$, *prove Euler's equation that*

$$\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf.$$

Exercise 149 *Given that* $x^2 + y^2 + z^2 = 6$ *and* $w^3 + z^3 = 5xy + 12$ *find*

$$\left(\frac{\partial z}{\partial x}\right)_y, \quad \left(\frac{\partial z}{\partial x}\right)_w, \quad \left(\frac{\partial z}{\partial y}\right)_x, \quad \left(\frac{\partial z}{\partial y}\right)_w, \quad \left(\frac{\partial w}{\partial x}\right)_z, \quad \left(\frac{\partial x}{\partial w}\right)_z$$

at the point $(x, y, z, w) = (1, -2, 1, 1)$

Exercise 150 *Given* $f(x, y, z) = 0$ *find expressions for*

$$\left(\frac{\partial y}{\partial x}\right)_z, \quad \left(\frac{\partial x}{\partial y}\right)_z, \quad \left(\frac{\partial y}{\partial z}\right)_x, \quad \left(\frac{\partial z}{\partial x}\right)_y.$$

Show that

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial x}\right)_z = 1 \quad \text{and} \quad \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$

8. PARTIAL DIFFERENTIAL EQUATIONS

8.1 Introduction

We have already solved, in earlier chapters, a few simple examples of partial differential equations (PDEs), namely:

- Example 88 (i): $f_{yx} = 0$ has solution $f(x, y) = A(x) + B(y)$,
- Example 88 (ii): $f_{xx} = 0$ has solution $f(x, y) = C(y)x + D(y)$,
- Example 103: $xf_x = yf_y$ has solution $f(x, y) = E(xy)$,

where A, B, C, D, E are arbitrary differentiable functions.

And we've already noted that it's not surprising to see a kth order partial differential equations having general solutions involving k arbitrary functions in the same way kth order ordinary differential equations often involve k indeterminate constants; this is because differential operators like $\partial/\partial x$ and $\partial/\partial y$ eliminate all functions which all functions not involving their variable. As with ordinary differential equations though it is not always the case that k functions will be involved.

In this first course in multivariate calculus we will not look to discuss more than a few important examples of PDEs and hardly touch on the problem of solving a PDE with boundary conditions (something for the Hilary term course). Here though we do give one method of finding *some* solutions to a particular PDE with boundary conditions.

Example 107 *Find all solutions of the form $T(x, t) = A(x)B(t)$ to the heat/diffusion equation*

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

where κ is a constant, in the range $0 \leqslant x \leqslant 1$, $0 \leqslant t$ and which satisfy the boundary conditions

$$T(0, t) = T(1, t) = 0 \quad \textit{for all } t \geqslant 0. \tag{8.1}$$

Solution. Solutions of the form $A(x)B(t)$ are known as **separable solutions** and most solutions will not be of this form. Nevertheless they do have an important role in solving the equation generally (see next term's PDE course). If we substitute

$$T(x, t) = A(x)B(t)$$

into the heat equation we find

$$A(x)B'(t) = \kappa A''(x)B(t)$$

and separating the variables we get

$$\frac{A''(x)}{A(x)} = \frac{B'(t)}{\kappa B(t)} = c. \tag{8.2}$$

Let's call the quantity in equation (8.2) $c(x, t)$. Note $c(x, t)$ is both a function solely of x (from the LHS) and a function solely of t (from the RHS). It follows that for any $(x_1, t_1), (x_2, t_2)$

$$\begin{aligned} c(x_1, t_1) &= c(x_1, t_2) && [\text{as } c \text{ only depends on } x] \\ &= c(x_2, t_2) && [\text{as } c \text{ only depends on } t]. \end{aligned}$$

That is, c is constant! We find the solutions in $A(x)$ from (8.2) depend now on whether c is positive, negative or zero. We have the following three possibilities:

$$A(x) = \begin{cases} P_1 \exp(\sqrt{c}x) + Q_1 \exp(-\sqrt{c}x) & c > 0, \\ P_2 x + Q_2 & c = 0, \\ P_3 \cos(\sqrt{-c}x) + Q_3 \sin(\sqrt{-c}x) & c < 0. \end{cases}$$

As we shall see only one of these cases can fit the boundary conditions $A(0) = A(1) = 0$, which follow from (8.1), in a non-zero fashion.

- Case (i): $c > 0$. We have $A(x) = P_1 \exp(\sqrt{c}x) + Q_1 \exp(-\sqrt{c}x)$. The boundary conditions imply

$$P_1 + Q_1 = 0, \qquad P_1 e^{\sqrt{c}} + Q_1 e^{-\sqrt{c}} = 0.$$

  Hence $P_1 = Q_1 = 0$ and $A(x) = 0$ is the only solution of this form to satisfy $A(0) = A(1) = 0$.

- Case (ii): $c = 0$. We have $A(x) = P_2 x + Q_2$. The boundary conditions imply that $P_2 = Q_2 = 0$ and, again, $A(x) = 0$ is the only solution of this form to satisfy $A(0) = A(1) = 0$.

- Case (iii): $c < 0$. We have $A(x) = P_3 \cos\left(\sqrt{-c}x\right) + Q_3 \sin\left(\sqrt{-c}x\right)$. The boundary conditions imply that

$$P_3 = 0, \qquad Q_3 \sin\left(\sqrt{-c}\right) = 0.$$

  If $A(x)$ is to be a non-zero solution we need Q_3 to be non-zero and hence $\sin\left(\sqrt{-c}\right) = 0$. This means $c = -n^2\pi^2$ for some non-zero integer n. For this value of c we have

$$\begin{aligned} A(x) &= Q_3 \sin(n\pi x), \\ B(t) &= \exp\left(-\kappa n^2 \pi^2 t\right) \quad \text{[up to a scalar multiple.]} \end{aligned}$$

  So we see there are non-zero separable solutions of the form

$$T(x,t) = K_n \sin(n\pi x) \exp\left(-\kappa n^2 \pi^2 t\right)$$

  where n is a non-zero integer and K_n is real.

■

Remark 108 Note that any sum of separable solutions will also solve the original heat equation, but generally this sum will not, itself, be a separable solution. So the separable solutions are far from being the only ones. However the separable solutions do have an important role in the general solution of the above boundary-value problem as the general solution can be written as an infinite sum of the separable solutions. See the Hilary Term courses on Partial Differential Equations and Fourier Series.

8.2 Laplace's Equation

In Cartesian co-ordinates **Laplace's equation** reads as

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= 0 \qquad \text{(in the plane);} \\ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} &= 0 \qquad \text{(in three dimensions).} \end{aligned}$$

A function which satisfies Laplace's equation is known as **harmonic**. Laplace's equation is often abbreviated to

$$\nabla^2 f = 0$$

and the differential operator ∇^2 is known as the **Laplacian.**

Laplace's equation appears in many physical situations. For example, the following are harmonic:

- the gravitational potential in a region containing no matter;
- the electrostatic potential in a charge-free region;

- the steady-state temperature in a region containing no source of heat;
- the velocity potential of an incompressible fluid with no vortices, sources or sinks.

Harmonic functions also have an important place in pure mathematics, for example in the study of complex functions.

Example 109 *Let $z = x + iy$ and let $f(z) = z^n$. If we write $u = \operatorname{Re} f$ and $v = \operatorname{Im} f$ so that*

$$u(x, y) + iv(x, y) = (x + iy)^n, \tag{8.3}$$

show that $u(x, y)$ and $v(x, y)$ are both harmonic functions.

Solution. If we differentiate (8.3) twice with respect to x, and also twice with respect to y, we get

$$\begin{aligned} u_{xx} + iv_{xx} &= n(n-1)(x+iy)^{n-2}; \\ u_{yy} + iv_{yy} &= n(n-1)i^2(x+iy)^{n-2} = -n(n-1)(x+iy)^{n-2}. \end{aligned}$$

Hence, adding the previous two equations,

$$(u_{xx} + u_{yy}) + i(v_{xx} + v_{yy}) = 0.$$

Comparing real and imaginary parts we see that u and v are both harmonic. ■
The above example extends more generally to differentiable complex functions — see Exercises 159 and 160.

As before we can look to find important classes of solutions of Laplace equation by insisting that the solutions have a certain form. For example we can consider separable solutions (see Exercise 156), circularly symmetric solutions (Example 110 below) and spherically symmetric solutions (see Exercise 163).

Example 110 *Show that Laplace's equation in planar polar co-ordinates reads as*

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0. \tag{8.4}$$

Hence find all circularly symmetric solutions to Laplace's equation in the plane — i.e. find all harmonic functions $f(r)$ which depend only on r and are independent of θ.

Solution. Firstly we need to know how Laplace's equation reads in polar co-ordinates. Recall that $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$ so that

$$\begin{aligned} r_x &= x\left(x^2+y^2\right)^{-1/2}, \qquad r_{xx} = \left(x^2+y^2\right)^{-1/2} - x^2\left(x^2+y^2\right)^{-3/2} = y^2/r^3, \\ \theta_x &= \left(-y/x^2\right)/\left(1+y^2/x^2\right) = -y/\left(x^2+y^2\right), \qquad \theta_{xx} = 2xy\left(x^2+y^2\right)^{-2}. \end{aligned}$$

Similarly

$$\begin{aligned} r_y &= y\left(x^2+y^2\right)^{-1/2}, \qquad r_{xx} = \left(x^2+y^2\right)^{-1/2} - y^2\left(x^2+y^2\right)^{-3/2} = x^2/r^3, \\ \theta_y &= (1/x)/\left(1+y^2/x^2\right) = x/\left(x^2+y^2\right), \qquad \theta_{yy} = -2xy\left(x^2+y^2\right)^{-2}. \end{aligned}$$

So for any f defined on the plane

$$\begin{aligned} f_{xx} + f_{yy} &= f_r(r_{xx} + r_{yy}) + f_\theta(\theta_{xx} + \theta_{yy}) + f_{rr}\left(r_x^2 + r_y^2\right) + 2f_{r\theta}(r_x\theta_x + r_y\theta_y) + f_{\theta\theta}\left(\theta_x^2 + \theta_y^2\right) \\ &= f_r\left(y^2+x^2\right)/r^3 + f_\theta(0) + f_{rr}\left(x^2+y^2\right)/r^2 + 2f_{r\theta}(0) + f_{\theta\theta}\left(x^2+y^2\right)/r^4 \\ &= f_r/r + f_{rr} + f_{\theta\theta}/r^2 \end{aligned}$$

Hence Laplace's equation, in terms of planar polar co-ordinates, reads as (8.4). We are interested in finding the circularly symmetric solutions, solutions which are independent of θ. In this case (8.4) now reads as

$$\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r} = 0.$$

This equation has integrating factor r and so we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right)=r\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}+\frac{\mathrm{d}f}{\mathrm{d}r}=0$$

giving

$$\frac{\mathrm{d}f}{\mathrm{d}r}=\frac{A}{r}$$

and so the general solution is

$$f(r)=A\ln r+B$$

■

One useful aspect of parabolic co-ordinates (see Example 93) is that Laplace's equation reads the same in both Cartesian and parabolic co-ordinates.

Example 111 *Recall the definition of parabolic planar co-ordinates*

$$x=\frac{u^2-v^2}{2},\qquad y=uv.$$

Show that Laplace's equation

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}=0$$

transforms into the same equation in parabolic co-ordinates.

Solution. From the second order chain rule we have

$$\begin{aligned}
F_{uu} &= F_x x_{uu}+F_y y_{uu}+F_{xx}(x_u)^2+2F_{xy}x_u y_u+F_{yy}(y_u)^2\\
&= F_x+u^2F_{xx}+2uvF_{xy}+v^2F_{yy}\\
F_{vv} &= F_x x_{vv}+F_y y_{vv}+F_{xx}(x_v)^2+2F_{xy}x_v y_v+F_{yy}(y_v)^2\\
&= -F_x+v^2F_{xx}-2uvF_{xy}+u^2F_{yy}.
\end{aligned}$$

Hence

$$F_{uu}+F_{vv}=\left(u^2+v^2\right)(F_{xx}+F_{yy})$$

and Laplace's equation reads the same in each co-ordinate system. ■

Example 112 *Let $A\in\mathbb{R}$ and $n\in\mathbb{Z}$. Show that the function $g(x,y)=A\sinh(n\pi x)\sin(n\pi y)$ satisfies Laplace's equation and the boundary conditions*

$$g(t,0)=g(t,1)=0\quad \textit{for } t\geqslant 0\quad \textit{and } g(0,t)=0\quad \textit{for}\quad 0\leqslant t\leqslant 1.$$

By using parabolic co-ordinates, or otherwise, find a non-zero solution $h(x,y)$ of Laplace's equation which satisfies the boundary conditions

$$h(t,0)=0\quad \textit{for}\quad t\geqslant 0,\quad h(t,1/t)=0\quad \textit{for } t\geqslant 1,\quad \textit{and}\quad h(t,t,)=0\quad \textit{for}\quad 0\leqslant t\leqslant 1.$$

Solution. Firstly note

$$g_{xx}=n^2\pi^2 A\sinh(n\pi x)\sin(n\pi y)\,,\quad\text{and}\quad g_{yy}=-n^2\pi^2 A\sinh(n\pi x)\sin(n\pi y)$$

and so $g(x,y)$ is indeed harmonic. Further

$$\begin{aligned}
g(t,0) &= A\sinh(n\pi t)\sin(0)=0,\\
g(t,1) &= A\sinh(n\pi t)\sin(n\pi)=0,\\
g(0,t) &= A\sinh(0)\sin(n\pi t)=0,
\end{aligned}$$

as required.

If we combine the previous part (with $A = n = 1$) and the previous example we know that the function

$$h(x,y) = \sinh \frac{\pi \left(x^2 - y^2\right)}{2} \sin(\pi xy)$$

satisfies Laplace's equation and also is zero when

$$\frac{x^2 - y^2}{2} = 0 \quad \text{or} \quad xy = 1 \quad \text{or} \quad xy = 0,$$

which, we see, include the given boundary conditions. ■

Poisson's equation is the inhomogeneous Laplace equation

$$\nabla^2 f = g.$$

It appears (amongst other places) in gravitational theory: Laplace's equation $\nabla^2 \phi = 0$ dictates how the *gravitational potential* ϕ behaves in the absence of any matter, but, more generally, when there is matter present,.distributed with density function $\rho(x,y,z)$, then Poisson's equation

$$\nabla^2 \phi = -4\pi G \rho$$

describes the potential's behaviour. As the Laplacian is linear then the only added complication in solving Poisson's equation — as compared with solving Laplace's equation — is in finding a particular solution

Example 113 *Find the circularly symmetric solutions of Poisson's equation in the plane $\nabla^2 f = g$ where*

$$g(r) = \begin{cases} r & \textit{for } r < a \\ 0 & \textit{for } r > a \end{cases}$$

Solution. Recall that the Laplacian is given by

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2}$$

in planar polar co-ordinates. We are only interested in circularly symmetric functions of the form $f(r)$; so for $r < a$ we need to solve

$$f''(r) + r^{-1} f'(r) = r.$$

If we multiply both sides by r we get

$$\left(r f'(r)\right)' = r f''(r) + f'(r) = r^2.$$

Hence

$$\begin{aligned} r f'(r) &= r^3/3 + A \Longrightarrow f'(r) = r^2/3 + A r^{-1} \\ f(r) &= r^3/9 + A \ln r + B. \end{aligned}$$

If we solve the equation similarly in the region $r > a$ then we find

$$f(r) = C \ln r + D.$$

There are several things to note though. Firstly if this function is to be defined at $r = 0$ it must be the case that $A = 0$. But also there are issues of continuity and smoothness at the $r = a$ border. For our solution f to be continuous at $r = a$ we need

$$a^3/9 + B = C \ln a + D$$

and for it to be smooth at $r = a$ (that is $\mathrm{d}f/\mathrm{d}r$ agrees on both sides of $r = a$) we need $a^2/3 = C/a$. Hence

$$A = 0, \qquad C = a^3/3, \qquad D = \left(a^3/9\right)(1 - 3\ln a) + B.$$

So the most general circularly symmetric solution of this Poisson equation is

$$f(r) = \begin{cases} \frac{r^3}{9} + B & \text{for } r \leqslant a, \\ \frac{a^3}{9} \ln r + \frac{a^3}{9}(1 - 3\ln a) + B & \text{for } r \geqslant a. \end{cases}$$

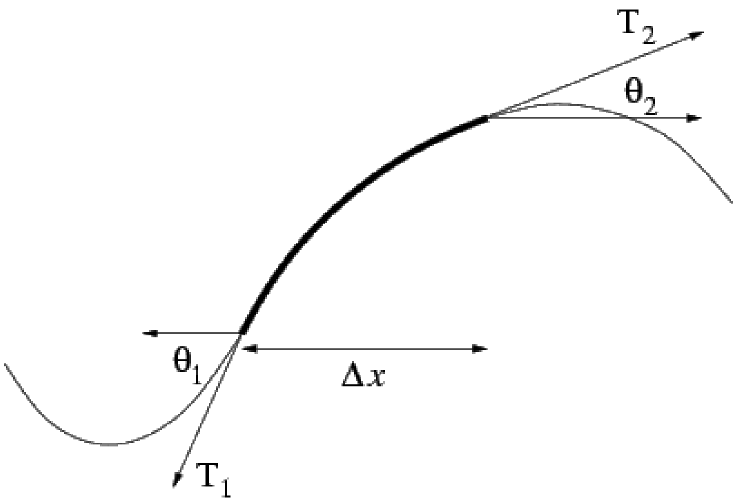
To specify the solution uniquely then we might, for example, require a further condition such as $f(0) = 0$ which would give $B = 0$. ■

8.3 The Wave Equation

We will consider the vibrations of an elastic uniform string under certain simplifying assumptions:-

- the string undergoes small vibrations: by this we mean that we shall consider second order terms, like y^2 and y_x^2 negligible;
- the vibrations are entirely *transverse*, so that a point at distance x_0 along the string remains on the line $x = a$ through out the motion;
- the string is at tension T and the density of the string is uniformly ρ;
- the effects of gravity and air resistance are negligible compared with the tension in the string.

Consider the vibrations of a small section of the string from x to $x + \Delta x$. We shall denote the tension of the string at $y(x,t)$ as T_1 and the tension at $y(x + \Delta x, t)$ as T_2, and likewise denote the angles the string makes with horizontal at those two point by θ_1 and θ_2 as shown in the diagram below.



If we resolve the forces in the x-direction we get that

$$T_2 \cos\theta_2 = T_1 \cos\theta_1$$

as the wave is vibrating transversely (i.e. this segment of string is only moving up and down), and taking components in the y-direction we get

$$T_2 \sin\theta_2 - T_1 \sin\theta_1 = (\rho\Delta x)\frac{\partial^2 y}{\partial t^2}(x_0, t)$$

where x_0 is the x-co-ordinate of the segment of mass, and because $\rho\Delta x$ is the segment's mass and y_{tt} its vertical acceleration. Note that x_0 will lie between x and $x + \Delta x$ as the entirety of the segment of string lies in this range. (Remember the weight of the string we consider negligible compared with the tension involved).

If we divide the previous equation by the constant $T_1 \cos\theta_1 = T_2 \cos\theta_2$ which, to the level of second order approximation we are making, equals the tension T, we get (with a little rearranging)

$$\frac{\tan\theta_2 - \tan\theta_1}{\Delta x} = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}(x_0, t).$$

Now

$$\tan\theta_1 = \frac{\partial y}{\partial x}(x,t), \qquad \tan\theta_2 = \frac{\partial y}{\partial x}(x + \Delta x, t)$$

and so by the Mean-Value Theorem (see next term's Analysis) it follows that

$$\frac{\partial^2 y}{\partial x^2}(c,t) = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}(x_0, t),$$

for some $c \in (x, x + \Delta x)$. Now letting Δx tend to 0, and assuming continuity of y_{xx} and y_{tt} we get

$$\frac{\partial^2 y}{\partial x^2}(x,t) = \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}(x,t).$$

Note that ρ/T has units $(kg/m)/(kgms^{-2}) = (s/m)^2$ so that $c = \sqrt{T/\rho}$ has the units of velocity. The wave-equation then reads

$$c^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}.$$

Theorem 114 *(D'Alembert 1746) The general solution of the wave equation*

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

is

$$y(x,t) = f(x - ct) + g(x + ct).$$

Proposition 115 ***Proof.*** *We introduce two new variables*

$$\zeta = x - ct \quad \text{and} \quad \eta = x + ct.$$

Then by Theorem 105 we have

$$\begin{aligned} y_{xx} &= y_\zeta \zeta_{xx} + y_\eta \eta_{xx} + y_{\zeta\zeta}(\zeta_x)^2 + 2y_{\zeta\eta}\zeta_x\eta_x + y_{\eta\eta}(\eta_x)^2 \\ &= y_\zeta \times 0 + y_\eta \times 0 + y_{\zeta\zeta} \times (1)^2 + 2y_{\zeta\eta} \times 1 \times 1 + y_{\eta\eta} \times (1)^2 \\ &= y_{\zeta\zeta} + 2y_{\zeta\eta} + y_{\eta\eta} \end{aligned}$$

and

$$\begin{aligned} y_{tt} &= y_\zeta \zeta_{tt} + y_\eta \eta_{tt} + y_{\zeta\zeta}(\zeta_t)^2 + 2y_{\zeta\eta}\zeta_t\eta_t + y_{\eta\eta}(\eta_t)^2 \\ &= y_\zeta \times 0 + y_\eta \times 0 + y_{\zeta\zeta} \times (-c)^2 + 2y_{\zeta\eta} \times (-c) \times c + y_{\eta\eta} \times (c)^2 \\ &= c^2(y_{\zeta\zeta} - 2y_{\zeta\eta} + y_{\eta\eta}) \end{aligned}$$

Hence $c^2 y_{xx} = y_{tt}$ *if*

$$c^2(y_{\zeta\zeta} + 2y_{\zeta\eta} + y_{\eta\eta}) = c^2(y_{\zeta\zeta} - 2y_{\zeta\eta} + y_{\eta\eta})$$

so that

$$y_{\zeta\eta} = 0.$$

We know, from Example 88(i), that the general solution of this is

$$y = f(\zeta) + g(\eta) = f(x - ct) + g(x + ct).$$

■

- Consider a solution of the form $y(x,t) = f(x - ct)$, with $g = 0$. Note that this solution at time $t + T$ resembles the solution at time t, but translated to the right by cT. This is a wave moving to the right with speed c and likewise a wave of the form $g(x + ct)$ is one which is moving to the left at speed c.

Example 116 *Find the solution to the wave equation when*

$$y(x,0) = \begin{cases} 1 - |x| & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}, \qquad \frac{\partial y}{\partial t}(x,0) = 0.$$

Sketch your solutions at $ct = 0, \frac{1}{2}, 1, \frac{3}{2}$.

Solution. We know that the solution has the form $y(x,t) = f(x-ct) + g(x+ct)$ and so the initial conditions give that

$$\begin{aligned} f(x) + g(x) &= \begin{cases} 1 - |x| & -1 < x < 1, \\ 0 & \text{otherwise,} \end{cases} \\ -cf'(x) + cg'(x) &= 0. \end{aligned}$$

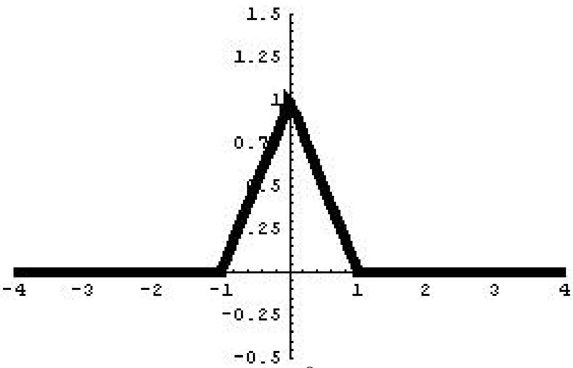
Integrating the second equation we see that $f(x) = g(x) + K$ where K is a constant. We can, without any loss of generality take K to be 0. [Keeping K in the calculation would yield to the same solution though with slightly different f and g.] So $f = g$ and we have

$$y(x,t) = f(x-ct) + f(x+ct)$$

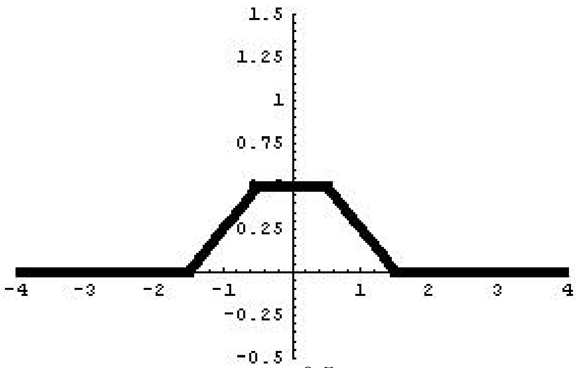
where

$$f(u) = \begin{cases} \frac{1}{2}(1 - |u|) & -1 < u < 1, \\ 0 & \text{otherwise.} \end{cases}$$

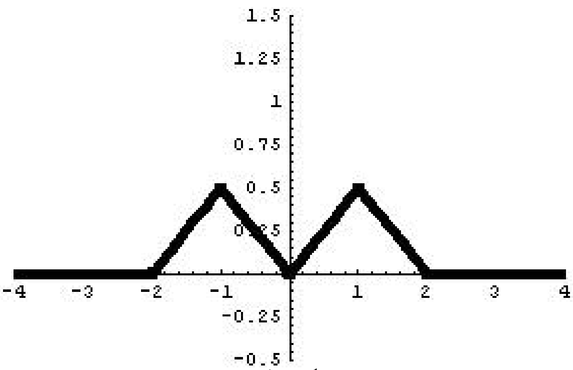
Sketches of the solution at different t are given below.



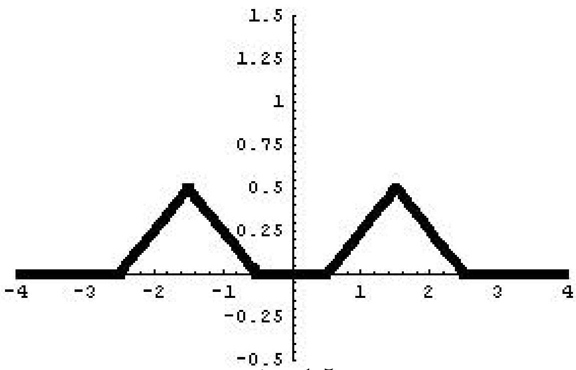
$t = 0$



$ct = 0.5$



$ct = 1$



$ct = 1.5$

■

8.4 Exercises

Exercise 151 *Show that the function* $f(x,y) = \tan^{-1}(y/x)$ *satisfies Laplace's equation in the plane. Find functions* g *and* h *such that* f *can be written in the form*

$$f(x,y) = g(x+iy) + h(x-iy).$$

Exercise 152 *By making the substitutions* $\eta = x - y$ *and* $\zeta = x - 2y$ *show that the general solution of*

$$2f_{xx} + 3f_{xy} + f_{yy} = 0$$

is $f(x,y) = A(x-y) + B(x-2y)$ *where* A *and* B *are arbitrary differentiable functions of one variable.*

Exercise 153 *Let* k *be an integer. Show that* $r^k \cos k\theta, r^k \sin k\theta$ *satisfy Laplace's equation in planar polar co-ordinates (See Equation (8.4)). Why is the requirement made that* k *is an integer?*

Hence find a solution to Laplace's equation which satisfies the boundary condition

$$f(1,\theta) = \cos^2\theta + \sin\theta.$$

Exercise 154 *Write the solutions* $r^k \cos k\theta$ *and* $r^k \sin k\theta$ *in the previous exercise in the form* $g(x+iy) + h(x-iy)$.

Exercise 155 *Either directly, or by using polar co-ordinates, show that if* $f(x,y)$ *is harmonic, then so is* $f\left(x/r^2, y/r^2\right)$ *where* $r^2 = x^2 + y^2$.

Exercise 156 *Show that separable solutions* $f(x,y) = X(x)Y(y)$ *of Laplace's equation in Cartesian co-ordinates take one of the forms*

$$\begin{aligned}
X(x) &= Ax + B, \qquad Y(y) = Cy + D, \\
X(x) &= A\exp\left(\sqrt{c}x\right) + B\exp\left(-\sqrt{c}x\right), \qquad Y(y) = C\cos\left(\sqrt{c}y\right) + D\sin\left(\sqrt{c}y\right) \quad \textit{where } c > 0, \\
X(x) &= A\cos\left(\sqrt{c}x\right) + B\sin\left(-\sqrt{c}x\right), \qquad Y(y) = C\exp\left(\sqrt{c}y\right) + D\exp\left(-\sqrt{c}y\right) \quad \textit{where } c > 0.
\end{aligned}$$

Exercise 157 *Show that separable solutions* $f(r,\theta) = R(r)\Theta(\theta)$ *to Laplace's equation in planar polar co-ordinates must satisfy the equations*

$$\begin{aligned}
r^2R'' + rR' - cR &= 0, \qquad (8.5) \\
\Theta'' + c\Theta &= 0,
\end{aligned}$$

for some constant c.

By making the substitution $r = e^t$ *and setting* $T(t) = R(r) = R(e^t)$ *into equation (8.5) show that*

$$\begin{aligned}
T(t) &= Ae^{kt} + Be^{-kt} \quad \textit{if } c = k^2 > 0, \\
T(t) &= At + B \quad \textit{if } c = 0, \\
T(t) &= A\sin(kt) + B\cos(kt) \quad \textit{if } c = -k^2 < 0.
\end{aligned}$$

Hence find all solutions of Laplace's equation which are separable in polar co-ordinates.

Exercise 158 *Let* $x \in \mathbb{R}$ *and* $t > 0$. *Verify that*

$$T(x,t) = \frac{A}{\sqrt{t}}\exp\left(\frac{-x^2}{4\kappa t}\right)$$

is a solution of the heat equation $\frac{\partial T}{\partial t} = \kappa\frac{\partial^2 T}{\partial x^2}$. *Sketch* T *as a function of* x *at two different times* t. *Show that*

$$\int_{-\infty}^{\infty} T(x,t)\, dx$$

is a constant function of t.

Exercise 159 ** Show that the real and imaginary parts of the following functions are harmonic:*

$$e^z = e^{x+iy} = e^x \cos x + ie^x \sin y; \qquad \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

Exercise 160 ** Let $f : \mathbb{C} \to \mathbb{C}$ be a complex-valued function with real and imaginary parts u and v so that*

$$f(x+iy) = u(x,y) + iv(x,y).$$

*f is said to be **differentiable** at $z \in \mathbb{C}$ if*

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists as $h \to 0$ in $\mathbb{C}$.

(i) Let $h = x \in \mathbb{R}$. Show that

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

(ii) Let $h = iy$ where $y \in \mathbb{R}$. Show that

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

*(iii) Deduce the **Cauchy-Riemann equations** which say that $u_x = v_y$ and $v_x = -u_y$. Hence show that u and v are harmonic functions.*

Exercise 161 *The co-ordinates u and v are defined by*

$$x = \cosh u \cos v, \qquad y = \sinh u \sin v$$

where x and y are Cartesian co-ordinates. Show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{1}{\sin^2 v + \sinh^2 u}\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right).$$

Exercise 162 *(i) Given spherical polar co-ordinates r, θ, ϕ show that*

$$\begin{aligned}
\mathbf{e}_r &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \\
\mathbf{e}_\phi &= (-\sin\phi, \cos\phi) \\
\mathbf{e}_\theta &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)
\end{aligned}$$

are mutually perpendicular unit vectors.

(ii) Show that if $\mathbf{r}(t)$ is a vector with co-ordinates $r(t), \phi(t), \theta(t)$ at time t then

$$\mathbf{r}'(t) = \dot{r}\mathbf{e}_r + \dot{\phi}(r\sin\theta\mathbf{e}_\phi) + \dot{\theta}(r e_\theta).$$

Exercise 163 *Show that the Laplacian $\nabla^2 f$ equals*

$$\frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{1}{r^2 \sin\theta}\frac{\partial^2 f}{\partial \phi^2}.$$

Hence find the spherically symmetric solutions of Laplace's equation in three dimensions.

Exercise 164 ** Let x_1, x_2 be co-ordinates describing two-dimensional space (e.g. Cartesian or polar).*

(i) Explain why there exist functions h_1, h_2 of x_1, x_2 and unit vectors $\mathbf{e}_1, \mathbf{e}_2$ such that for any path $\mathbf{r}(t)$ in the plane we have

$$\mathbf{r}'(t) = h_1 \dot{x}_1 \mathbf{e}_1 + h_2 \dot{x}_2 \mathbf{e}_2.$$

*(ii) If the vectors $\mathbf{e}_1, \mathbf{e}_2$ are orthogonal then x_1, x_2 are said to be **orthogonal curvilinear co-ordinates**. Show that Cartesian, polar and parabolic co-ordinates are all orthogonal curvilinear and find $h_1, h_2, \mathbf{e}_1, \mathbf{e}_2$ in each case*

Exercise 165 ** Show that, in terms of orthogonal curvilinear co-ordinates x_1, x_2 in the plane*

$$\nabla^2 f = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{h_2}{h_1}\frac{\partial f}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_1}{h_2}\frac{\partial f}{\partial x_2}\right)\right].$$

Verify that this formula agrees with (8.4) in the case of planar polar co-ordinates.

Exercise 166 ** Let $L > 0$. Find all the separable solutions*

$$y(x,t) = X(x)\,T(t) \tag{8.6}$$

of the wave equation $y_{tt} = c^2 y_{xx}$ which satisfies the boundary conditions

$$y(0,t) = y(L,t) = 0 \quad \text{for all } t.$$

Exercise 167 ** A string with small vertical displacement $y(x,t)$, where $a \leqslant x \leqslant b$, is fixed at each end so that $y(a,t) = 0 = y(b,t)$ for all time t. Show that the string's energy*

$$E(t) = \int_a^b \left\{\frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 + \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2\right\} dx$$

is constant throughout the motion. (The first term of the energy formula is the tensile energy and the second term represents the string's kinetic energy.)

Exercise 168 ** Recall that the general solution of the wave equation $y_{tt} = c^2 y_{xx}$ has the form $y(x,t) = f(x+ct) + g(x-ct)$. If further we impose the initial conditions*

$$y(x,0) = h(x), \qquad y_t(x,0) = v(x)$$

show that

$$y(x,t) = \frac{h(x+ct) + h(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} v(s)\,\mathrm{d}s.$$

Exercise 169 ** A stretched string is fixed at two end-points $x = 0$ and $x = l$ so that $y(0,t) = 0 = y(l,t)$ for all times t. Investigate the natural modes of vibration of the string by seeking solutions to the wave equation of the form*

$$y(x,t) = f(x)\sin\omega t.$$

Show that, with the given boundary conditions, such vibrations can occur only at certain natural frequencies

$$\frac{\omega}{2\pi} = \frac{nc}{2l}, \qquad n = 1, 2, 3, \ldots$$

Show that the higher the value of n, the larger the number of nodes (a node is a point of zero displacement on the string throughout its motion).

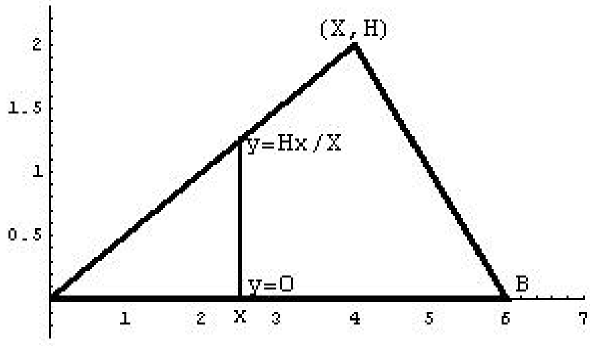
9. JACOBIANS AND AREA

9.1 A Brief Introduction to Double Integrals

In a moment we will meet the definition of a Jacobian. The definition is simple enough, but, in order to motivate the idea behind it, we will need to calculate some areas. As we change from one co-ordinate system to another the equations of curves change, the co-ordinates of points, the rules for functions. Likewise the Jacobian is a rule for how measuring areas, or volumes, changes with a co-ordinate change. Our first two examples of calculating areas involves two examples of *double integrals.*

Example 117 *Calculate the area of the triangle with vertices* $(0,0),(B,0)$ *and* (X,H).

Solution.



We have known, since early school days that the answer is $\frac{1}{2}BH$, but we shall demonstrate this here by means of a double integral. The three bounding lines of the triangles are

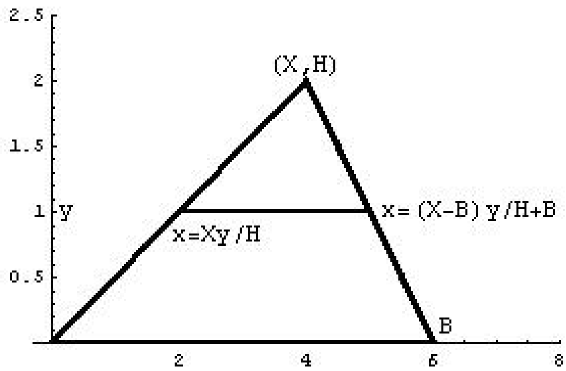
$$y=0,\qquad y=\frac{H}{X}x,\qquad y=\frac{H}{X-B}(x-B).$$

We'll assume here that $0<X<B$. In order to "pick up" all of the triangle's area we need to let x range from 0 to B and y range appropriately from 0 up to the bounding line above $(x,0)$. Note that the equation for this bounding line changes at $x=X$ which is why we need the split the integral in two in the calculation below. As our x and y vary over the triangle we need to pick up an infinitesimal piece of area $\mathrm{d}x\ \mathrm{d}y$ at each point. We can then calculate the triangle's area as

$$\begin{aligned}
A &= \int_{x=0}^{x=X}\int_{y=0}^{y=Hx/X}\mathrm{d}y\,\mathrm{d}x+\int_{x=X}^{x=B}\int_{y=0}^{y=H(x-B)/(X-B)}\mathrm{d}y\,\mathrm{d}x\\
&= \int_{x=0}^{x=X}\frac{Hx}{X}\,\mathrm{d}x+\int_{x=X}^{x=B}\frac{H(x-B)}{X-B}\,\mathrm{d}x\\
&= \frac{H}{X}\left[\frac{x^2}{2}\right]_0^X+\frac{H}{X-B}\left[\frac{(x-B)^2}{2}\right]_X^B\\
&= \frac{H}{X}\frac{X^2}{2}-\frac{H}{X-B}\frac{(X-B)^2}{2}\\
&= \frac{HX}{2}-\frac{H(X-B)}{2}=\frac{HB}{2}.
\end{aligned}$$

Integrating this way, y first then x, we would need to treat $X<0$ and $X>B$ as two further cases. Alternatively, one could also pick up the area by first letting y range from 0 to H and letting x range over the

interior points of the triangle at height y.



This method is somewhat better as we don't have to treat the three cases of $X < 0$, $0 < X < B$ and $B < X$ separately.

$$
\begin{aligned}
A &= \int_{y=0}^{y=H} \int_{x=Xy/H}^{x=y(X-B)/H+B} \mathrm{d}x \, \mathrm{d}y \\
&= \int_{y=0}^{y=H} \left(\frac{(X-B)\,y}{H} + B - \frac{yX}{H} \right) \mathrm{d}y \\
&= \int_{y=0}^{y=H} \left(B - \frac{By}{H} \right) \mathrm{d}y \\
&= B \left[y - \frac{y^2}{2H} \right]_{y=0}^{y=H} \\
&= B \left(H - \frac{H^2}{2H} \right) = \frac{BH}{2}
\end{aligned}
$$

■

Example 118 *Calculate the area of the disc $x^2 + y^2 \leqslant a^2$.*

Solution. Again we know the answer, namely πa^2. If we wish to pick up all of the disc's area we can let x vary over the range $-a$ to a and, at each x, we need to let y vary from $-\sqrt{a^2 - x^2}$ to $\sqrt{a^2 - x^2}$. So we have

$$
\begin{aligned}
A &= \int_{x=-a}^{x=a} \int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} \mathrm{d}y \, \mathrm{d}x \\
&= \int_{x=-a}^{x=a} 2\sqrt{a^2 - x^2} \; \mathrm{d}x \\
&= \int_{\theta=-\pi/2}^{\theta=\pi/2} 2\sqrt{a^2 - a^2 \sin^2\theta}\, a \cos\theta \, \mathrm{d}\theta \qquad [x = a \sin\theta] \\
&= a^2 \int_{-\pi/2}^{\pi/2} 2\cos^2\theta \, \mathrm{d}\theta \\
&= a^2 \int_{-\pi/2}^{\pi/2} 1 + \cos 2\theta \; \mathrm{d}\theta \\
&= a^2 \left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\pi/2}^{\pi/2} = \pi a^2.
\end{aligned}
$$

■

In the first example of the triangle using y as the outside variable and x the inside helped avoid making special cases. For this example of the disc though it would have been much more natural to use polar co-ordinates — *if* we knew how to calculate areas with such!

9.2 Jacobians

You may be aware that the modulus of a determinant of a 2×2 matrix A is a measure of how much the map $\mathbf{x} \longmapsto A\mathbf{x}$ stretches area. In this case the map does so uniformly at each point. The Jacobian, or rather its modulus, is a measure of how a map stretches space locally, near a particular point, when this stretching effect varies from point to point. The Jacobian takes its name from the German mathematician Carl Jacobi (1804-1851).

Definition 119 *Given two co-ordinates $u(x,y)$ and $v(x,y)$ which depend on variables x and y, we define the* ***Jacobian***

$$\frac{\partial(u,v)}{\partial(x,y)}$$

to be the determinant

$$\det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

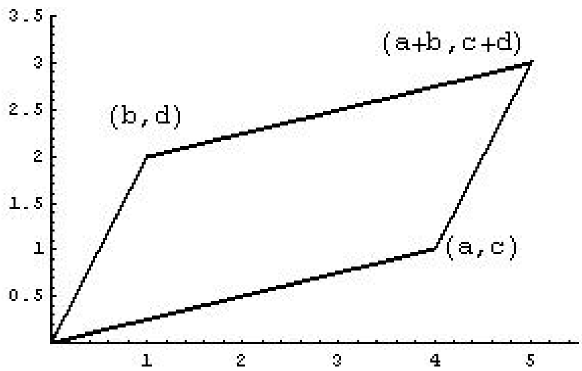
Example 120 *Let u and v be related by the equation*

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}. \tag{9.1}$$

Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad-bc.$$

Consider the parallelogram that is the image under the above map (9.1) of the square with vertices $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$. This should have area $|ad-bc|$.



The area of a parallelogram with sides $\mathbf{a}$ *and* $\mathbf{b}$ *is* $|\mathbf{a} \wedge \mathbf{b}|$ *Hence the above parallelogram's area is*

$$|(a\mathbf{i}+c\mathbf{j}) \wedge (b\mathbf{i}+d\mathbf{j})| = |ad\mathbf{i} \wedge \mathbf{j} + bc\mathbf{j} \wedge \mathbf{i}| = |(ad-bc)\,\mathbf{k}| = |ad-bc|.$$

More generally any region with area A will transform under the map (9.1) to a region with area $|ad-bc| \times A$.

Example 121 *Let $x = r\cos\theta$ and $y = r\sin\theta$ where r and θ are polar co-ordinates. Then*

$$\begin{aligned} \frac{\partial(x,y)}{\partial(r,\theta)} &= \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \\ &= \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\ &= r\left(\cos^2\theta + \sin^2\theta\right) = r. \end{aligned}$$

Example 122 *In reverse,* $r=\sqrt{x^2+y^2}$ *and* $\theta=\tan^{-1}(y/x)$ *and*

$$\begin{aligned}\frac{\partial(r,\theta)}{\partial(x,y)} &= \det\begin{pmatrix}\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y}\end{pmatrix}\\ &= \det\begin{pmatrix}\frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}}\\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2}\end{pmatrix}\\ &= \frac{x^2+y^2}{(x^2+y^2)^{3/2}}\\ &= \frac{1}{\sqrt{x^2+y^2}}=\frac{1}{r}\end{aligned}$$

The Jacobian satisfies its own sort of chain rule as one might expect, given that the effect on area of two scalings would be the product of the scalings.

Proposition 123 *Let* r *and* s *be functions of variables* u *and* v *which in turn are functions of* x *and* y. *Then*

$$\frac{\partial(r,s)}{\partial(x,y)}=\frac{\partial(r,s)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)}$$

Proof.

$$\begin{aligned}\frac{\partial(r,s)}{\partial(x,y)} &= \det\begin{pmatrix}\frac{\partial r}{\partial x} & \frac{\partial r}{\partial y}\\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y}\end{pmatrix}\\ &= \det\begin{pmatrix}\frac{\partial r}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial r}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial r}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial r}{\partial v}\frac{\partial v}{\partial y}\\ \frac{\partial s}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial s}{\partial v}\frac{\partial v}{\partial x} & \frac{\partial s}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial s}{\partial v}\frac{\partial v}{\partial y}\end{pmatrix}\\ &= \det\left\{\begin{pmatrix}\frac{\partial r}{\partial u} & \frac{\partial r}{\partial v}\\ \frac{\partial s}{\partial u} & \frac{\partial s}{\partial v}\end{pmatrix}\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}\right\}\\ &= \frac{\partial(r,s)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)}.\end{aligned}$$

■

Corollary 124

$$\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(x,y)}=1$$

Solution. Take $r=x$ and $s=y$ in the previous proposition. Indeed the stronger result holds true that

$$\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}=I_2.$$

■

Example 125 *Recall the definition of parabolic planar co-ordinates*

$$x=\frac{1}{2}\left(u^2-v^2\right),\qquad y=uv.$$

In this case the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ *is given by*

$$\begin{aligned}\frac{\partial(x,y)}{\partial(u,v)} &= \det\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}\\ &= \det\begin{pmatrix}u & -v\\ v & u\end{pmatrix}\\ &= u^2+v^2.\end{aligned}$$

Though it is somewhat messy to calculate u and v in terms of x and y we can easily calculate that

$$\begin{aligned}
\left(u^2+v^2\right)^2 &= u^4+2u^2v^2+v^4 \\
&= \left(u^2-v^2\right)^2+4u^2v^2 \\
&= 4x^2+4y^2
\end{aligned}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{u^2+v^2} = \frac{1}{4\left(x^2+y^2\right)}.$$

Example 126 *Calculate u_x, u_y, v_x, v_y in terms of u and v where*

$$x = a\cosh u\cos v, \qquad y = a\sinh u\sin v.$$

Solution. We have

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} = \begin{pmatrix} a\sinh u\cos v & -a\cosh u\sin v \\ a\cosh u\sin v & a\sinh u\cos v \end{pmatrix}.$$

Hence

$$\begin{aligned}
\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} &= \begin{pmatrix} a\sinh u\cos v & -a\cosh u\sin v \\ a\cosh u\sin v & a\sinh u\cos v \end{pmatrix}^{-1} \\
&= \frac{1}{a\left(\sinh^2 u\cos^2 v+\cosh^2 u\sin^2 v\right)}\begin{pmatrix} \sinh u\cos v & \cosh u\sin v \\ -\cosh u\sin v & \sinh u\cos v \end{pmatrix}.
\end{aligned}$$

■

Though 3×3 determinants are off this syllabus, (see Hilary term Linear Algebra) included here are the Jacobians of spherical and cylindrical polar co-ordinates.

Example 127 *(Cylindrical Polar Co-ordinates)*

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

Then

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(r,\theta,z)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial r} \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} \\
&= r
\end{aligned}$$

Example 128 *(Spherical Polar Co-ordinates)*

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta.$$

Then

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\phi} & \frac{\partial x}{\partial\theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\phi} & \frac{\partial y}{\partial\theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\phi} & \frac{\partial z}{\partial\theta} \end{vmatrix} \\
&= \begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} \\
&= \cos\theta\begin{vmatrix} -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ r\sin\theta\cos\phi & r\cos\theta\sin\phi \end{vmatrix} - r\sin\theta\begin{vmatrix} \sin\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi \end{vmatrix} \\
&= r^2\cos\theta\sin\theta\cos\theta\left(-\sin^2\phi-\cos^2\phi\right) - r^2\sin^3\theta\left(\cos^2\phi+\sin^2\phi\right) \\
&= -r^2\sin\theta\left(\cos^2\theta+\sin^2\theta\right) \\
&= -r^2\sin\theta.
\end{aligned}$$

Definition 129 *Let $R \subseteq \mathbb{R}^2$. Then we define the* ***area*** *of R to be*

$$A(R) = \int\int_{(x,y)\in R} \mathrm{d}x\,\mathrm{d}y.$$

Quite what this definition formally means, (given that integrals will not be formally defined until Trinity term analysis), and for what regions of R it makes sense to talk of their "area", is actually a very complicated question. But for the simple examples we shall give this definition will be clear and their area unambiguous.

Theorem 130 *Let $f : R \to S$ be a bijection between two regions of $\mathbb{R}^2$, and write $(u,v) = f(x,y)$. Suppose that*

$$\frac{\partial(u,v)}{\partial(x,y)}$$

is defined and non-zero everywhere. Then

$$A(S) = \int\int_{(u,v)\in S} du\,dv = \int\int_{(x,y)\in R} \left|\frac{\partial(u,v)}{\partial(x,y)}\right| dx\,dy$$

Example 131 *If we return to the case of calculating the area of the disc $x^2+y^2 \leqslant a^2$, now using polar co-ordinates we have a much simpler double integral. The interior of the disc, in polar co-ordinates, is given by*

$$0 \leqslant r \leqslant a, \qquad 0 \leqslant \theta < 2\pi.$$

So

$$\begin{aligned}
A &= \int\int_{x^2+y^2\leqslant a^2} \mathrm{d}x\,\mathrm{d}y = \int_{r=0}^{r=a}\int_{\theta=0}^{\theta=2\pi} \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| \mathrm{d}\theta\,\mathrm{d}r \\
&= \int_{r=0}^{r=a}\int_{\theta=0}^{\theta=2\pi} r\,\mathrm{d}\theta\,\mathrm{d}r \\
&= \int_{r=0}^{r=a} [\theta r]_{\theta=0}^{\theta=2\pi}\ \mathrm{d}r \\
&= 2\pi\int_{r=0}^{r=a} r\,\mathrm{d}r \\
&= 2\pi\left[\frac{r^2}{2}\right]_{r=0}^{r=a} = \pi a^2.
\end{aligned}$$

Proof. (Sketch proof of preceding theorem) Consider the small element of area that is bounded by the co-ordinate lines $u = u_0$ and $u = u_0 + \delta u$ and $v = v_0$ and $v = v_0 + \delta v$. Let's say that $f(x_0,y_0) = (u_0,v_0)$ and consider small changes δx and δy in x and y respectively. We have a slightly distorted parallelogram with sides

$$\begin{aligned}
\mathbf{a} &= f(x_0+\delta x, y_0) - f(x_0,y_0) \approx \frac{\partial f}{\partial x}(x_0,y_0)\,\delta x, \\
\mathbf{b} &= f(x_0, y_0+\delta y) - f(x_0,y_0) \approx \frac{\partial f}{\partial y}(x_0,y_0)\,\delta y,
\end{aligned}$$

ignoring higher order terms in δx and δy. As f takes values in $\mathbb{R}^2$ then the above are vectors in $\mathbb{R}^2$. The area of a parallelogram in $\mathbb{R}^2$ with sides $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a}\wedge\mathbf{b}|$ where $\wedge$ denotes the vector product. So the element of area we are considering is (ignoring higher order terms)

$$\left|\frac{\partial f}{\partial x}\delta x \wedge \frac{\partial f}{\partial y}\delta y\right| = \left|\frac{\partial f}{\partial x}\wedge\frac{\partial f}{\partial y}\right|\delta x\,\delta y.$$

Now $f = (u,v)$, so $f_x = (u_x, v_x)$, $f_y = (u_y, v_y)$ and

$$\frac{\partial f}{\partial x}\wedge\frac{\partial f}{\partial y} = \left(\frac{\partial u}{\partial x},\frac{\partial v}{\partial x}\right)\wedge\left(\frac{\partial u}{\partial y},\frac{\partial v}{\partial y}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)\mathbf{k}.$$

Finally

$$\begin{aligned} \delta A &\approx \left|\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right| \delta x\, \delta y. \\ &= \left|\frac{\partial (u,v)}{\partial (x,y)}\right| \delta x\, \delta y. \end{aligned}$$

■

Example 132 *A* ***shear*** *parallel to the x-axis is a map of the form*

$$(x,y) \longmapsto (x+ky,y).$$

Note that this can also be written as

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

and that the determinant of the above matrix is 1, so that shears are area-preserving.

If we consider the effect of the shear

$$\begin{pmatrix} 1 & -X/H \\ 0 & 1 \end{pmatrix}$$

on the co-ordinates $(0,0),(X,H),(B,0)$ of the vertices of the triangle in Example 117 we see that they are moved to $(0,0),(0,H)$,$(B,0)$. So we see that we can calculate the general triangle's area by just considering the special case when $X=0$.

Indeed, if we were so inclined, we could note that the ***stretch***

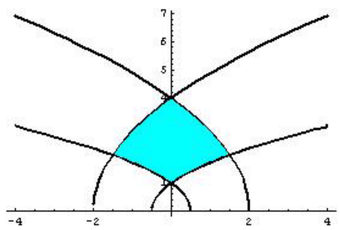
$$\begin{pmatrix} 1/B & 0 \\ 0 & 1/H \end{pmatrix}$$

has determinant $1/(BH)$ and moves the triangle to $(0,0),(1,0),(0,1)$. Hence the original triangle has area BH times this simpler triangle.

Example 133 *Calculate the area bounded by the curves*

$$2x=1-y^2, \qquad 2x=y^2-1, \qquad 8x=16-y^2, \qquad 8x=y^2-16.$$

as shown in the diagram below.



Solution. We see, if we change to planar polar co-ordinates $x=(u^2-v^2)/2$, $y=uv$, that the region in question is $1\leqslant u\leqslant 2$, $1\leqslant v\leqslant 2$. Hence the area is given by

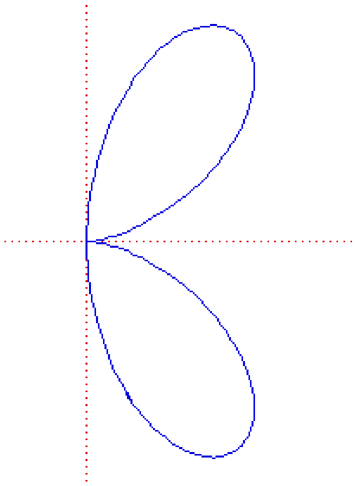
$$\begin{aligned} A &= \int_{u=1}^{u=2}\int_{v=1}^{v=2}\left|\frac{\partial (x,y)}{\partial (u,v)}\right| \mathrm{d}v\, \mathrm{d}u \\ &= \int_{u=1}^{u=2}\int_{v=1}^{v=2}\left(u^2+v^2\right)\, \mathrm{d}v\, \mathrm{d}u \\ &= \int_{u=1}^{u=2}\left[u^2v+\frac{v^3}{3}\right]_{v=1}^{v=2} \mathrm{d}u \\ &= \int_{u=1}^{u=2}\left(u^2+\frac{7}{3}\right) \mathrm{d}u \\ &= \left[\frac{u^3}{3}+\frac{7u}{3}\right]_1^2=\frac{7}{3}+\frac{7}{3}=\frac{14}{3}. \end{aligned}$$

■

Example 134 *The area of the sector bounded by the half-lines $\theta = \alpha$ and $\theta = \beta$ and the polar curve $r = R(\theta)$ equals*

$$\begin{aligned} A &= \int_{\theta=\alpha}^{\theta=\beta} \int_{r=0}^{r=R(\theta)} r \, \mathrm{d}r \, \mathrm{d}\theta \\ &= \int_{\theta=\alpha}^{\theta=\beta} \left[\frac{r^2}{2} \right]_0^{R(\theta)} \mathrm{d}\theta \\ &= \frac{1}{2} \int_{\theta=\alpha}^{\theta=\beta} R(\theta)^2 \, \mathrm{d}\theta \end{aligned}$$

Example 135 *Calculate the area bounded by the **double folium** (see diagram) which has Cartesian equation* $\left(x^2 + y^2\right)^2 = 4axy^2$.



Solution. If we convert to polar co-ordinates this becomes

$$r = 4a \cos\theta \sin^2\theta$$

and so the area equals bounded by the two leaves equals

$$\begin{aligned} A &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} 16a^2 \cos^2\theta \sin^4\theta \, \mathrm{d}\theta \\ &= 16a^2 \int_0^{\pi/2} \sin^4\theta - \sin^6\theta \, \mathrm{d}\theta \\ &= 16a^2 \left(I_4 - I_6\right), \end{aligned}$$

where $I_n = \int_0^{\pi/2} \sin^n\theta \, \mathrm{d}\theta$. Now

$$\begin{aligned} I_{n+2} &= \int_0^{\pi/2} \sin^{n+1}\theta \sin\theta \, \mathrm{d}\theta \\ &= \left[-\sin^{n+1}\theta \cos\theta\right]_0^{\pi/2} + \int_0^{\pi/2} (n+1) \sin^n\theta \cos^2\theta \, \mathrm{d}\theta \\ &= (n+1)\left(I_n - I_{n+2}\right). \end{aligned}$$

Hence $I_{n+2} = (n+1) I_n / (n+2)$ and so

$$I_0 = \frac{\pi}{2}; \qquad I_2 = \frac{\pi}{4}; \qquad I_4 = \frac{3\pi}{16}; \qquad I_6 = \frac{15\pi}{96}.$$

Finally then

$$A = 16a^2 \left(I_4 - I_6\right) = 16a^2 \times \left(\frac{3\pi}{16} - \frac{15\pi}{96}\right) = \pi a^2 \left(3 - \frac{15}{6}\right) = \frac{\pi a^2}{2}.$$

■

Example 136 *To calculate the volume of the sphere $x^2 + y^2 + z^2 \leqslant a^2$, it seems most natural to use spherical polar co-ordinates. The interior of the sphere, in spherical polar co-ordinates, is given by*

$$0 \leqslant r \leqslant a, \qquad 0 \leqslant \phi < 2\pi, \qquad 0 \leqslant \theta < \pi.$$

So

$$\begin{aligned}
A &= \iint\int_{x^2+y^2+z^2 \leqslant a^2} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{r=0}^{r=a}\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\left|\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)}\right| \mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}r \\
&= \int_{r=0}^{r=a}\int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi} r^2 \sin\theta\ \mathrm{d}\theta\,\mathrm{d}\phi\ \mathrm{d}r \\
&= \left[\frac{r^3}{3}\right]_{r=0}^{r=a} [\phi]_{\phi=0}^{\phi=2\pi} [-\cos\theta]_{\theta=0}^{\theta=\pi} \\
&= \frac{a^3}{3} \times 2\pi \times 2 = \frac{4\pi a^3}{3}.
\end{aligned}$$

Example 137 ** Equal Area Projections*

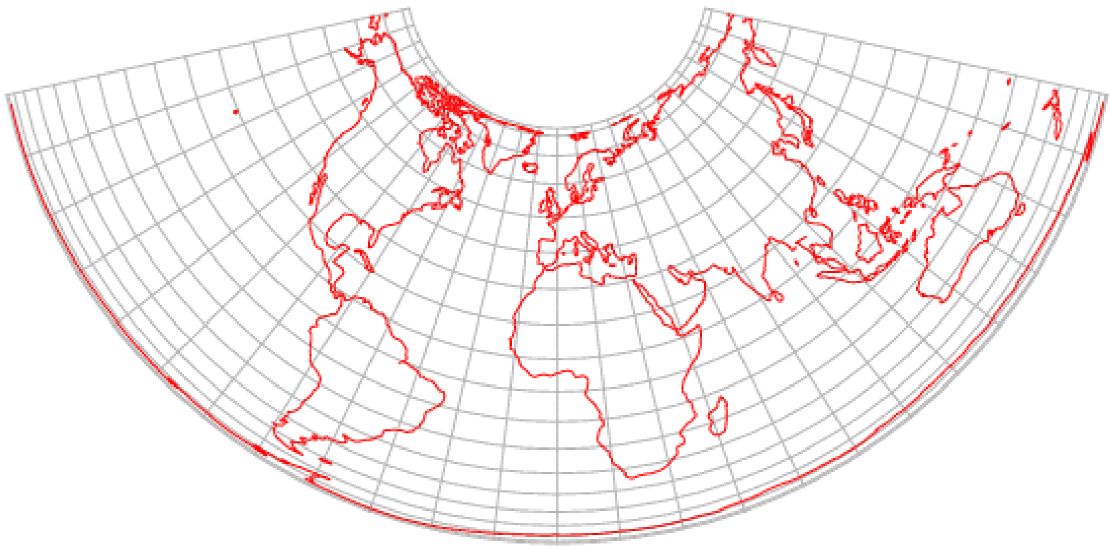
Recall the definition of spherical polar co-ordinates

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$

which has Jacobian

$$\left|\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)}\right| = r^2\sin\theta$$

where $\frac{\pi}{2} - \theta$ is latitude and ϕ is longitude. If then we are considering the element of area on the planet's surface $r = a$, represented by θ and ϕ, then the local area distortion is $a^2\sin\theta$.



The Albers Equal-area Conic Projection is an equal-area projection. In terms of θ and ϕ the projection is

$$x = \frac{1}{n}\sqrt{C - 2n\cos\theta}\sin n\phi, \qquad y = \rho_0 - \frac{1}{n}\sqrt{C - 2n\cos\theta}\cos n\phi,$$

where n, C, ρ_0 are constants. Then the Jacobian of the projection is

$$\begin{aligned}
\left|\frac{\partial(x,y)}{\partial(\phi,\theta)}\right| &= \left|\det\begin{pmatrix} \sqrt{C - 2n\cos\theta}\cos n\phi & \frac{\sin\theta}{\sqrt{C-2n\cos\theta}}\sin n\phi \\ \sqrt{C - 2n\cos\theta}\sin n\phi & -\frac{2n\sin\theta}{\sqrt{C-2n\cos\theta}}\cos n\phi \end{pmatrix}\right| \\
&= \cos^2 n\phi\sin\theta + \sin^2 n\phi\sin\theta \\
&= \sin\theta,
\end{aligned}$$

so that equal areas on the map represent equal areas of the planet.

9.3 Exercises

Exercise 170 *Let* $u(x,y) = xy$ *and* $v(x,y) = y/x$. *Calculate the Jacobian* $\partial(u,v)/\partial(x,y)$. *Show that the area in the first quadrant* $\{(x,y) : x > 0, y > 0\}$ *bounded by the curves*

$$y = 2x, \qquad 2y = x, \qquad xy = 2, \qquad xy = 8,$$

equals $6 \ln 2$.

Exercise 171 *Calculate the Jacobians* $\frac{\partial(u,v)}{\partial(x,y)}$ *and* $\frac{\partial(x,y)}{\partial(u,v)}$, *and verify that* $\frac{\partial(u,v)}{\partial(x,y)}\frac{\partial(x,y)}{\partial(u,v)} = 1$ *in each of the following cases: (i)* $u = x + y$, $v = y/x$; *(ii)* $u = x^2/y$, $v = y^2/x$.

Exercise 172 *The variables* u *and* v *are given by*

$$u = x^2 - xy, \qquad v = y^2 + xy,$$

for all real x, y. *Find the partial derivatives* u_x, u_y, x_u, y_u *in terms of* x *and* y *stating the values of* x *and* y *for which your results are valid.*

Exercise 173 *Given that* u, v, w *are functions of the independent variables* x, y, z *for which*

$$u + v + w = x, \quad vw + wu + uv = y, \quad uvw = z,$$

find the partial derivative $\partial u/\partial x$ *in terms of* u, v, w *only.*

Exercise 174 * *Given orthogonal curvilinear co-ordinates as defined in Exercise 164 show that*

$$\frac{\partial(x,y)}{\partial(x_1,x_2)} = h_1 h_2.$$

Exercise 175 *Show that the area of the region bounded by circular arcs* $r = R$ *and* $r = R + \delta R$ *and the rays* $\theta = \alpha$ *and* $\theta = \alpha + \delta\alpha$ *equals*

$$R\,\delta R\,\delta\alpha$$

ignoring third order terms in $\delta\alpha$ *and* δR.

Exercise 176 *Evaluate*

$$\int_0^1 \int_0^{3-3x} \mathrm{d}y\,\mathrm{d}x, \qquad \int_0^2 \int_{y/2}^1 \mathrm{d}x\,\mathrm{d}y, \qquad \int_0^4 \int_0^{\sqrt{x}} \mathrm{d}y\,\mathrm{d}x, \qquad \int_0^1 \int_0^{\sqrt{1-y^2}} \mathrm{d}x\,\mathrm{d}y.$$

In each case sketch the area of integration.

Exercise 177 * *Prove that the double integrals*

$$\int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\,\mathrm{d}y \right) \mathrm{d}x, \qquad \int_0^1 \left(\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2}\,\mathrm{d}x \right) \mathrm{d}y$$

exist and evaluate them.

Exercise 178 *Calculate directly the area of the quadrilateral with vertices* $(1,3), (3,3), (2,6), (6,6)$. *Perform this calculation with both possible orders of Cartesian co-ordinates.*

Exercise 179 *Let* $a > 0$. *The curve with polar equation*

$$r = a(1 + \cos\theta), \qquad 0 \leqslant \theta < 2\pi,$$

is called a cardioid. Sketch the curve and show that the area bounded by it equals $\frac{3}{2}\pi a^2$.

Exercise 180 *Let $a > 0$. Sketch the curve*

$$x^{2/3} + y^{2/3} = a^{2/3}$$

in the quadrant $x, y > 0$.

The variables u and v are given in terms of x, y by

$$x = u\cos^3 v, \qquad y = u\sin^3 v.$$

What is the equation of the curve in terms of the new co-ordinates u and v? Calculate the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ and hence find the area of the region bounded by the curve and the positive x- and y-axes.

Exercise 181 *By considering an appropriate stretch find the area of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Likewise, find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Exercise 182 *Calculate the double integral*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y$$

via a change to polar co-ordinates and deduce that

$$\int_{-\infty}^{\infty} e^{-s^2}\,\mathrm{d}s = \sqrt{\pi}.$$

Exercise 183 ** The* ***mean value*** *of a function f defined on a region V is given by the formula*

$$\mu = \frac{1}{\mathrm{Vol}(V)}\int_V f.$$

Find the mean value of the function $xyz(x^2 + y^2 + z^2)$ in the region V given by

$$V = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, 0 \leq x \leq y, -1 \leq z \leq 1\}.$$

How would you define the ***median*** *of a function f defined on V?*

Exercise 184 ** Evaluate the integral*

$$\iint\limits_A \left(x^2 + y^2\right)^2\,\mathrm{d}x\,\mathrm{d}y$$

where A is the region in the positive quadrant enclosed by the curves

$$x^3 - 3xy^2 = a, \quad x^3 - 3xy^2 = b, \quad 3x^2y - y^3 = c, \quad 3x^2y - y^3 = d$$

where $0 < a < b$ and $0 < c < d$.

Exercise 185 ** Find, using polar co-ordinates the centre of mass of a uniform quadrant of a circle of radius a.*

Exercise 186 ** Find the volume of the region satisfying the inequalities*

$$x^2 + y^2 + z^2 \leqslant a^2, \qquad h \leqslant z \leqslant a.$$

Hence find the height of the centre of mass of a uniform solid hemisphere of radius a.

Exercise 187 ** Let $b > a > 1$. Consider integrating the function e^{-xy} over the range $(0, \infty) \times (a, b)$ in two different ways; assuming the order of integration does not matter show that*

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \, dx = \log\left(\frac{b}{a}\right).$$

Exercise 188 ** The Bessel function $J_0(x)$ can be defined for $x \in \mathbb{R}$ as*

$$J_0(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \cos\theta) \, \mathrm{d}\theta.$$

Assuming the order of integration does not matter, show that for $p > 0$,

$$\int_0^\infty J_0(x) \, e^{-px} \, \mathrm{d}x = \frac{1}{\sqrt{1+p^2}}.$$

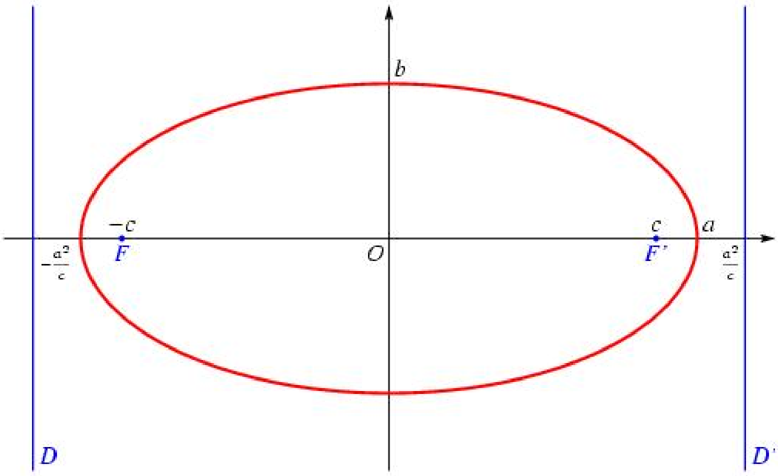
10. THE GRADIENT VECTOR

10.1 Standard Curves and Surfaces

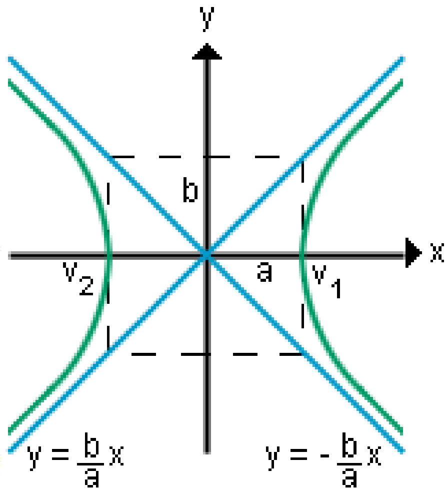
In the next section we will meet the *gradient vector* ∇f of a scalar function f. The gradient vector is a way of calculating the normal of a curve, or surface, $f = c$ (where c is a constant) defined by a function. Here we introduce a few standard examples of some curves and surfaces

Example 138 *(Conics) The standard form of equation of the conics (circle, ellipse, parabola, hyperbola) are*

- ***circle:*** $x^2 + y^2 = a^2$; *parametrisation* $(a\cos\theta, a\sin\theta)$; *area* πa^2.
- ***ellipse:*** $x^2/a^2 + y^2/b^2$ $(b < a)$; *parametrisation* $(a\cos\theta, b\sin\theta)$; *eccentricity* $e^2 = 1 - b^2/a^2$; *foci* $(\pm ae, 0)$; *directrices* $x = \pm a/e$; *area* πab.
- ***parabola:*** $y^2 = 4ax$; *parametrisation* $\left(at^2, 2at\right)$; *eccentricity* $e = 1$; *focus* $(a, 0)$; *directrix* $x = -a$.
- ***hyperbola:*** $x^2/a^2 - y^2/b^2 = 1$; *parametrisation* $(a\sec t, b\tan t)$ *or* $(\pm a\cosh t, b\sinh t)$; *eccentricity* $e^2 = 1 + b^2/a^2$; *foci* $(\pm ae, 0)$; *directrices* $x = \pm a/e$; *asymptotes* $y = \pm bx/a$



ellipse



hyperbola

Example 139 *Find the point of the ellipse* $x^2/4 + y^2/9 = 1$ *which is closest to the point* $\left(5/\sqrt{2}, 5/\sqrt{2}\right)$.

Solution. In order to solve this we can turn the question into a one-variable maximisation problem by parametrises the ellipse as $(2\cos t, 3\sin t)$ The distance $D(t)$ of this point from $\left(5/\sqrt{2}, 5/\sqrt{2}\right)$ is given by

$$D(t)^2 = \left(\frac{5}{\sqrt{2}} - 2\cos t\right)^2 + \left(\frac{5}{\sqrt{2}} - 3\sin t\right)^2.$$

If we differentiate this we get

$$2D\frac{\mathrm{d}D}{\mathrm{d}t} = 2\left(\frac{5}{\sqrt{2}} - 2\cos t\right)(2\sin t) + 2\left(\frac{5}{\sqrt{2}} - 3\sin t\right)(-3\cos t) = 0.$$

This rearranges to

$$\sqrt{2}\sin t = \cos t\left(\frac{3}{\sqrt{2}} - \sin t\right).$$

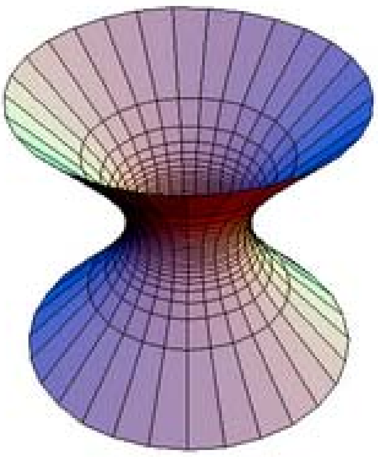
By inspection we see that when $t = \pi/4$ and $\sin t = \cos t = 1/\sqrt{2}$ the equation holds and the closest point is

$$\left(2\cos\frac{\pi}{4}, 3\sin\frac{\pi}{4}\right) = \left(\sqrt{2}, \frac{3}{\sqrt{2}}\right).$$

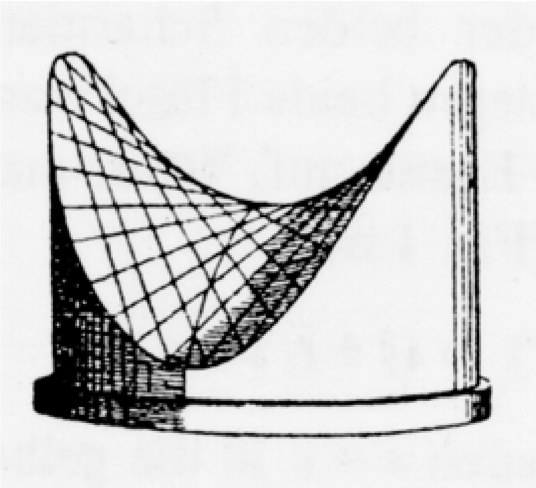
■

Example 140 *(Examples of Quadrics) The standard form of equation of the quadrics are*

- ***Sphere:*** $x^2 + y^2 + z^2 = a^2$ *;*
- ***Ellipsoid:*** $x^2/a^2 + y^2/b^2 + z^2/c^2$*;*
- ***Hyperboloid of One Sheet:*** $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$*;*
- ***Hyperboloid of Two Sheets:*** $x^2/a^2 - y^2/b^2 - z^2/c^2 = 1$*;*
- ***Paraboloid:*** $z = x^2 + y^2$*;*
- ***Hyperbolic Paraboloid:*** $z = x^2 - y^2$*;*
- ***Cone:*** $x^2 + y^2 = z^2$*.*



one sheet hyperboloid



hyperbolic paraboloid

Example 141 *Find the tangent and normal to the ellipse* $x^2/a^2 + y^2/b^2 = 1$ *at the point* (X, Y)*.*

Solution. One way would be to implicitly differentiate the equation and obtain

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

giving $\mathrm{d}y/\mathrm{d}x = -b^2X/\left(a^2Y\right)$ at the point in question. Then the normal has gradient $a^2Y/\left(b^2X\right)$. And so the two equations are

$$\begin{aligned} y - Y &= \frac{-b^2X}{a^2Y}(x - X) \qquad \text{(tangent)} \\ y - Y &= \frac{a^2Y}{b^2X}(x - X) \qquad \text{(normal)} \end{aligned}$$

The tangent's equation simplifies somewhat to

$$b^2Xx + a^2Yy = a^2b^2.$$

Alternatively we know that a parametrisation of the ellipse is given by

$$\mathbf{r}(t) = (a\cos t, b\sin t).$$

So a **tangent vector** to the ellipse at $\mathbf{r}(t)$ equals

$$\mathbf{r}'(t) = (-a\sin t, b\cos t).$$

This is a vector in the tangent direction. Hence a vector in the normal direction is

$$(b\cos t, a\sin t) = \left(\frac{bX}{a}, \frac{aY}{b}\right) = \frac{ba}{2}\left(\frac{2X}{a^2}, \frac{2Y}{b^2}\right).$$

Quite why the last term has been written that way will become clearer once we have met the gradient vector. ■

We probably all feel we know a smooth surface in $\mathbb{R}^3$ when we see one, and this instinct for what a surface is will be satisfactory for the purposes of this course. We recognise all the previous examples as smooth surfaces, with the exception of the cone at $(0,0,0)$ and could well be comfortable working out their tangent planes and normals. In the main we will be happy to work with surfaces without a formal definition, but for those seeking a more rigorous treatment of the topic, the following might provide a suitable working definition.

Definition 142 ** A **smooth parametrised surface** is a map $\mathbf{r}$, known as the **parametrisation***

$$\mathbf{r} : U \to \mathbb{R}^3 : (u,v) \mapsto (x(u,v), y(u,v), z(u,v))$$

*from an **open subset** $U \subseteq \mathbb{R}^2$ to $\mathbb{R}^3$ such that*

- *x, y, z have continuous partial derivatives with respect to u and v of all orders*
- *$\mathbf{r}$ is a bijection with both $\mathbf{r}$ and $\mathbf{r}^{-1}$ being continuous*
- *at each point the vectors*
$$\frac{\partial \mathbf{r}}{\partial u} \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v}$$
are linearly independent (i.e. are not scalar multiples of one another).

We will not be looking to treat this definition with any generality. Rather we shall just look to parametrise some of the "standard" surfaces previously described and calculate some tangents and normals. We define:

Definition 143 *Let $\mathbf{r} : U \to \mathbb{R}^3$ be a smooth parametrised surface and let $\mathbf{p}$ be a point on the surface. The plane containing $\mathbf{p}$ and which is parallel to the vectors*

$$\frac{\partial \mathbf{r}}{\partial u}(\mathbf{p}) \quad \text{and} \quad \frac{\partial \mathbf{r}}{\partial v}(\mathbf{p})$$

*is called the **tangent plane** to $\mathbf{r}(U)$ at $\mathbf{p}$. Because these vectors are independent the tangent plane is well-defined.*

Definition 144 *Any vector in the direction*

$$\frac{\partial \mathbf{r}}{\partial u}(\mathbf{p}) \wedge \frac{\partial \mathbf{r}}{\partial v}(\mathbf{p})$$

*is said to be **normal** to the surface at $\mathbf{p}$. There are two **unit normals** of length one.*

Example 145 *We have already met a parametrisation for the sphere $x^2 + y^2 + z^2 = a^2$ with spherical polar co-ordinates. This given by*

$$\mathbf{r}(\phi, \theta) = (a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta).$$

We already know the outward-pointing unit normal at $\mathbf{r}(\theta,\phi)$ *is* $\mathbf{r}(\theta,\phi)/a$ *but let's verify this with the previous definitions and find the tangent plane. We have*

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial \phi} &= (-a\sin\theta\sin\phi, a\sin\theta\cos\phi, 0),\\ \frac{\partial \mathbf{r}}{\partial \theta} &= (a\cos\theta\cos\phi, a\cos\theta\sin\phi, -a\sin\theta).\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial \phi}\wedge\frac{\partial \mathbf{r}}{\partial \theta} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ -a\sin\theta\sin\phi & a\sin\theta\cos\phi & 0\\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta\end{vmatrix}\\ &= a^2\sin\theta\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\sin\phi & \cos\phi & 0\\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\end{vmatrix}\\ &= a^2\sin\theta\begin{pmatrix}-\sin\theta\cos\phi\\ -\sin\theta\sin\phi\\ -\cos\theta\end{pmatrix}\end{aligned}$$

and so the two unit normals are $\pm(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$. *The tangent plane at* $\mathbf{r}(\phi,\theta)$ *is then*

$$\mathbf{r}\cdot(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)=a.$$

Example 146 *Find the normal and tangent plane to the point* (X,Y,Z) *on the hyperbolic paraboloid* $z=x^2-y^2$.

Solution. This surface has a simple choice of parametrisation as there is exactly one point lying above, or below, the point $(x,y,0)$. So we can take a parametrisation

$$\mathbf{r}(x,y)=\left(x,y,x^2-y^2\right).$$

We then have

$$\frac{\partial \mathbf{r}}{\partial x}=(1,0,2x),\qquad \frac{\partial \mathbf{r}}{\partial y}=(0,1,-2y).$$

So

$$\frac{\partial \mathbf{r}}{\partial x}\wedge\frac{\partial \mathbf{r}}{\partial y}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 2x\\ 0 & 1 & -2y\end{vmatrix}=\begin{pmatrix}-2x\\ 2y\\ 1\end{pmatrix}.$$

A normal vector to the surface at (X,Y,Z) is then $(-2X,2Y,1)$ and we see that the equation of the tangent plane is

$$\begin{aligned}\mathbf{r}\cdot(-2X,2Y,1) &= \left(X,Y,X^2-Y^2\right)\cdot(-2X,2Y,1)\\ &= -2X^2+2Y^2+X^2-Y^2\\ &= Y^2-X^2=-Z\end{aligned}$$

or equivalently

$$2Xx-2Yy+z=Z.$$

■

10.2 The Gradient Vector

In practice though, for the curves and surfaces we met in the last section, there will be no need to go through the rather laborious task of parametrising surfaces to find normals. This is because all these curves and surfaces we've seen are examples of *level sets* and the *gradient vector* is a simple way of finding normals to such surfaces.

Definition 147 *Given a scalar function $f : \mathbb{R}^n \to \mathbb{R}$, whose partial derivatives all exist, the* ***gradient vector***

$$\nabla f \quad \textit{or} \quad \operatorname{grad} f$$

is defined as

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right).$$

The symbol ∇ is pronounced "grad" usually, but also as "del" or "nabla".

Example 148 *Let*

$$f(x, y, z) = 2xy^2 + ze^x + yz.$$

Then

$$\nabla f = \left(2y^2 + ze^x, 4xy + z, e^x + y\right).$$

Example 149 *Let*

$$g(x, y, z) = x^2 + y^2 + z^2.$$

Then

$$\nabla g = (2x, 2y, 2z).$$

Note that ∇g is normal to the spheres $g =$ const.

Example 150 *Consider the vector field*

$$\mathbf{v}(x, y, z) = \left(2xy + z\cos(zx), x^2 + e^{y-z}, x\cos(zx) - e^{y-z}\right).$$

Find all the scalar functions $f(x, y, z)$ such that $\mathbf{v} = \nabla f$.

Solution. For any such f we have

$$\frac{\partial f}{\partial x} = 2xy + z\cos(zx)$$

and hence

$$f(x, y, z) = x^2 y + \sin(zx) + g(y, z),$$

for some function g of two variables. Now $\partial f/\partial y = x^2 + e^{y-z}$ and differentiating the above gives

$$x^2 + \frac{\partial g}{\partial y} = x^2 + e^{y-z}$$

so that

$$g(y, z) = e^{y-z} + h(z).$$

Finally $\partial f/\partial z = x\cos(zx) - e^{y-z}$ and differentiating the above with respect to z gives

$$x\cos(zx) - e^{y-z} + h'(z) = x\cos(zx) - e^{y-z}$$

so that $h'(z) = 0$ and h is a constant. So the given function f has the form

$$f(x, y, z) = x^2 y + \sin(zx) + e^{y-z} + \text{const.}$$

■

The formula for ∇ in planar polar co-ordinates is as follows:-

Proposition 151 *(Planar Polar Co-ordinates) Let*

$$\mathbf{e}_r = (\cos\theta, \sin\theta), \qquad \mathbf{e}_\theta = (-\sin\theta, \cos\theta).$$

These are perpendicular unit vectors, pointing along the curves $r =$ const. and $\theta =$ const. Further

$$\nabla \phi = \frac{\partial \phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\mathbf{e}_\theta.$$

Proof. It is easy to check that

$$\begin{aligned}
\mathbf{e}_r \cdot \mathbf{e}_r &= \cos^2\theta + \sin^2\theta = 1, \\
\mathbf{e}_\theta \cdot \mathbf{e}_\theta &= \sin^2\theta + \cos^2\theta = 1, \\
\mathbf{e}_r \cdot \mathbf{e}_\theta &= -\sin\theta\cos\theta + \sin\theta\cos\theta = 0.
\end{aligned}$$

The line $\theta = c$ is parametrised by $\theta(r) = (r\cos c, r\sin c)$ and we have $\theta'(r) = \mathbf{e}_r$; similarly $r(\theta) = (a\cos\theta, a\sin\theta)$ and we see $r'(\theta) = a\mathbf{e}_\theta$. Recall that the relationship between r, θ and x, y are given by

$$\begin{aligned}
r &= \sqrt{x^2+y^2}, & \frac{\partial r}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} = \cos\theta, & \frac{\partial r}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} = \sin\theta, \\
\theta &= \tan^{-1}\frac{y}{x}, & \frac{\partial \theta}{\partial x} &= \frac{-y}{x^2+y^2} = \frac{-\sin\theta}{r}, & \frac{\partial \theta}{\partial y} &= \frac{x}{x^2+y^2} = \frac{\cos\theta}{r},
\end{aligned}$$

and so

$$\begin{aligned}
\nabla\phi &= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} \\
&= \left(\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial x}\right)\mathbf{i} + \left(\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial y}\right)\mathbf{j} \\
&= \frac{\partial\phi}{\partial r}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + \frac{\partial\phi}{\partial\theta}\left(\frac{-\sin\theta}{r}\mathbf{i} + \frac{\cos\theta}{r}\mathbf{j}\right) \\
&= \frac{\partial\phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\mathbf{e}_\theta.
\end{aligned}$$

■

Definition 152 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable scalar function and let $\mathbf{u}$ be a unit vector. Then the **directional derivative** of f at $\mathbf{a}$ in the direction $\mathbf{u}$ equals*

$$\lim_{t\to 0}\left(\frac{f(\mathbf{a}+t\mathbf{u}) - f(\mathbf{a})}{t}\right).$$

This is the rate of change of the function f at $\mathbf{a}$ in the direction $\mathbf{u}$.

Example 153 *Let $f(x,y,z) = x^2y - z^2$ and let $\mathbf{a} = (1,1,1)$. Calculate the directional derivative of f at $\mathbf{a}$ in the direction $\mathbf{u} = (u_1, u_2, u_3)$. In what direction does f increase most rapidly?*

Solution. Now

$$\begin{aligned}
\frac{f(\mathbf{a}+t\mathbf{u}) - f(\mathbf{a})}{t} &= \frac{\left[(1+tu_1)^2(1+tu_2) - (1+tu_3)^2\right] - (1^2 1 - 1^2)}{t} \\
&= (2u_1 + u_2 - 2u_3) + \left(u_1^2 + 2u_1u_2\right)t + u_1^2u_2t^2 \\
&\to 2u_1 + u_2 - 2u_3 \quad \text{as } t \to 0.
\end{aligned}$$

This is the directional derivative asked for. What is the largest this can be as we vary $\mathbf{u}$ over all possible unit vectors? Well in the direction $\left(\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}\right)$ the directional derivative is

$$\frac{4}{3} + \frac{1}{3} + \frac{4}{3} = 3.$$

On the other hand

$$2u_1 + u_2 - 2u_3 = (2, 1, -2) \cdot \mathbf{u} = 3|\mathbf{u}|\cos\theta = 3\cos\theta$$

(where θ is the angle between $\mathbf{u}$ and $(2,-1,2)$) takes a maximum of 3 when $\theta = 0$ and $\mathbf{u} = (2/3, 1/3, -2/3)$. ■

Proposition 154 *The directional derivative of a function f at the point $\mathbf{a}$ in the direction $\mathbf{u}$ equals $\nabla f(\mathbf{a}) \cdot \mathbf{u}$*

Proof. Let

$$F(t) = f(\mathbf{a}+t\mathbf{u}) = f(a_1 + tu_1, \ldots, a_n + tu_n).$$

Then

$$\lim_{t \to 0}\left(\frac{f(\mathbf{a}+t\mathbf{u}) - f(\mathbf{a})}{t}\right) = \lim_{t \to 0}\left(\frac{F(t) - F(0)}{t}\right) = F'(0).$$

Now, by the chain rule,

$$\begin{aligned}
F'(0) &= \left.\frac{\mathrm{d}F}{\mathrm{d}t}\right|_{t=0} \\
&= \frac{\partial f}{\partial x_1}(\mathbf{a})\frac{\mathrm{d}x_1}{\mathrm{d}t} + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{a})\frac{\mathrm{d}x_n}{\mathrm{d}t} \\
&= \frac{\partial f}{\partial x_1}(\mathbf{a})\, u_1 + \cdots + \frac{\partial f}{\partial x_n}(\mathbf{a})\, u_n \\
&= \nabla f(\mathbf{a}) \cdot \mathbf{u}.
\end{aligned}$$

■

Corollary 155 *The rate of change of f is greatest in the direction ∇f, that is when $\mathbf{u} = \nabla f / |\nabla f|$.*

Definition 156 *A* ***level set*** *of a function $f : \mathbb{R}^3 \to \mathbb{R}$ is a set of points*

$$\left\{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = c\right\}$$

where c is a constant. For suitably "nice" functions f and constants c the level set is a surface in $\mathbb{R}^3$. Note that all the quadrics given in Example 140 are level sets.

Proposition 157 *Given a surface $S \subseteq \mathbb{R}^3$ with equation $f(x, y, z) = c$ and a point $\mathbf{p} \in S$ then $\nabla f(\mathbf{p})$ is normal to S at $\mathbf{p}$ equals.*

Proof. Let u and v be co-ordinates near $\mathbf{p}$ and $\mathbf{r} : (u, v) \to \mathbf{r}(u, v)$ be a parametrisation of part of S. Recall that the normal to S at $\mathbf{p}$ is in the direction

$$\frac{\partial \mathbf{r}}{\partial u} \wedge \frac{\partial \mathbf{r}}{\partial v}.$$

Note $f((\mathbf{r}(u, v))) = c$ and so $\partial f / \partial u = \partial f / \partial v = 0$. If we write $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$ then we see that

$$\begin{aligned}
\nabla f \cdot \frac{\partial \mathbf{r}}{\partial u} &= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \\
&= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u} \\
&= \frac{\partial f}{\partial u} \qquad \text{[by the chain rule]} \\
&= 0
\end{aligned}$$

as f is constant as a function of u and v. Similarly $\nabla f \cdot \partial \mathbf{r} / \partial v = 0$ and hence ∇f is in the direction of $\partial \mathbf{r} / \partial u \wedge \partial \mathbf{r} / \partial v$ and so normal to the surface. ■

Example 158 *In Example 146 we determined the normal at (X, Y, Z) to the hyperbolic paraboloid $z = x^2 - y^2$. If we introduce the function*

$$f(x, y, z) = x^2 - y^2 - z$$

then we can know the normal is

$$\nabla f(X, Y, Z) = (2X, -2Y, -1)$$

which is find the normal to vector we found previously.

Example 159 *Find the point on the ellipsoid*

$$\frac{x^2}{4}+\frac{y^2}{9}+z^2=1$$

which is closest to the plane $x+2y+z=10$.

Solution. The normal to the plane is parallel to $(1,2,1)$ everywhere. The point on the ellipsoid which is closest will also have normal $(1,2,1)$. If we set

$$f(x.y,z)=\frac{x^2}{4}+\frac{y^2}{9}+z^2-1,$$

then the gradient equals

$$\nabla f=\left(\frac{x}{2},\frac{2y}{9},2z\right).$$

If this is parallel to $(1,2,1)$ then we $x=2\lambda$, $y=9\lambda$, $z=\lambda/2$ for some λ. This point lies on the ellipsoid when

$$\frac{(2\lambda)^2}{4}+\frac{(9\lambda)^2}{9}+\left(\frac{\lambda}{2}\right)^2=1$$

giving $41\lambda^2/4=1$, and so $\lambda=2/\sqrt{41}$. The closest point then is

$$\left(\frac{4}{\sqrt{41}},\frac{18}{\sqrt{41}},\frac{1}{\sqrt{41}}\right).$$

■

Example 160 *The temperature* T *in* $\mathbb{R}^3$ *is given by*

$$T(x,y,z)=x+y^2-z^2.$$

Given that heat flows in the direction of $-\nabla T$, *describe the curve along which heat moves from the point* $(1,1,1)$.

Solution. We have

$$-\nabla T=(-1,-2y,2z),$$

which is the direction in which heat flows. If we parametrise the flow of the heat (say by arc-length s) as $\mathbf{r}(s)=(x(s),y(s),z(s))$ then we have

$$\frac{y'(s)}{y(s)}=2x'(s),\qquad \frac{z'(s)}{z(s)}=-2x'(s),$$

as (x',y',z') and $(-1,-2y,2z)$ are parallel vectors. Integrating the last two equations we get

$$\begin{aligned}\ln y(s) &= 2x(s)-2\\ \ln z(s) &= -2x(s)+2.\end{aligned}$$

Hence the equation of the heat's path is

$$2x=\ln y+2=2-\ln z,$$

moving along the direction of $(1,2,-2)$ from $(1,1,1)$. ■

As you might expect ∇ satisfies a selection of product, quotient and chain rules:

Proposition 161 *Let* f *and* g *be functions of* x,y,z. *Then*

$$\begin{array}{ll}\text{(i)} & \nabla(fg)= f\nabla g+g\nabla f\\ \text{(ii)} & \nabla(f^n)= nf^{n-1}\nabla f\\ \text{(iii)} & \nabla\left(\frac{f}{g}\right)= \dfrac{g\nabla f-f\nabla g}{g^2}\\ \text{(iv)} & \nabla(f\circ g)(\mathbf{x})= f'(g(\mathbf{x}))\nabla g(\mathbf{x})\end{array}$$

Proof. See Exercise 201 for (i), (ii), (iii). The proof of (iv) follows from the chain rule:-

$$\begin{aligned}\nabla (f \circ g)(\mathbf{x}) &= \left(\frac{\partial}{\partial x} f(g(\mathbf{x})), \frac{\partial}{\partial y} f(g(\mathbf{x})), \frac{\partial}{\partial z} f(g(\mathbf{x}))\right) \\ &= \left(f'(g(\mathbf{x}))\frac{\partial g}{\partial x}(\mathbf{x}), f'(g(\mathbf{x}))\frac{\partial g}{\partial y0}(\mathbf{x}), f'(g(\mathbf{x}))\frac{\partial g}{\partial z}(\mathbf{x})\right) \\ &= f'(g(\mathbf{x}))\nabla g(\mathbf{x})\end{aligned}$$

■

Example 162 ** There are two important operations related to grad ∇. For a vector function $\mathbf{v} = (v_1, v_2, v_3)$ we define*

$$\nabla \wedge \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

This is also written $\mathrm{curl}\mathbf{v}$*.*

Secondly

$$\nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

which is also written $\mathrm{div}\mathbf{v}$*.*

Example 163 *Perhaps unsurprisingly these operations satisfy the identities*

$$\begin{aligned}\nabla \cdot (\nabla \wedge \mathbf{v}) &= 0 \\ \nabla \wedge \nabla f &= \mathbf{0}\end{aligned}$$

for any vector function $\mathbf{v}$ and scalar function f. To prove the latter we see

$$\begin{aligned}\nabla \wedge \nabla f &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\ &= (f_{zy} - f_{yz})\mathbf{i} + (f_{xz} - f_{zx})\mathbf{j} + (f_{yx} - f_{xy})\mathbf{k} = \mathbf{0}.\end{aligned}$$

The Laplacian ∇^2 given by

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

can also be rewritten as

$$\nabla^2 f = \nabla \cdot \nabla f = \mathrm{div}\,(\mathrm{grad} f)$$

which explains its notation as ∇^2.

Example 164 ** Recall in Example 150 we introduced*

$$\mathbf{v}(x,y,z) = \left(2xy + z\cos(zx), x^2 + e^{y-z}, x\cos(zx) - e^{y-z}\right)$$

and noted that $\mathbf{v} = \nabla f$ where

$$f(x,y,z) = x^2 y + \sin(zx) + e^{y-z}.$$

From the previous example it follows that $\mathrm{curl}\mathbf{v} = \mathbf{0}$*. However, such scalars don't always exist; for example there is no f such that $\nabla f = \mathbf{w}$ where*

$$\mathbf{w} = (y, z, x).$$

Note that

$$\nabla \wedge \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = (-1,-1,-1) \neq \mathbf{0}.$$

In fact, the converse of $\nabla \wedge \nabla f = \mathbf{0}$ does hold in $\mathbb{R}^3$. That is, if $\mathbf{F}$ is a vector-valued function on $\mathbb{R}^3$ such that $\operatorname{curl}\mathbf{F} = \mathbf{0}$ *then there exists a scalar function ϕ, known as a **potential** such that $\mathbf{F} = \nabla\phi$.*

The potential ϕ can be defined as

$$\phi(\mathbf{p}) = \int_0^1 \mathbf{F}(t\mathbf{p}) \cdot \mathbf{p}\, \mathrm{d}t.$$

So in the case of $\mathbf{v}$ this equals

$$\begin{aligned}
& \int_0^1 \left\{ \left[2t^2xy + tz\cos\left(t^2zx\right)\right] x + \left[t^2x^2 + e^{t(y-z)}\right] y + \left[tx\cos\left(t^2zx\right) - e^{t(y-z)}\right] z \right\} \mathrm{d}t \\
= & \left[\frac{2t^3}{3}x^2y + \frac{1}{2}\sin\left(t^2zx\right) + \frac{t^3}{3}x^2y + \frac{ye^{t(y-z)}}{y-z} + \frac{1}{2}\sin\left(t^2zx\right) - \frac{ze^{t(y-z)}}{y-z}\right]_0^1 \\
= & x^2y + \sin(zx) + e^{y-z}.
\end{aligned}$$

10.3 Exercises

Exercise 189 ** Find the equations in Cartesian co-ordinates of:*
(i) the ellipse with foci at $(\pm 2, 0)$ which passes through $(0, 1)$;
(ii) the hyperbola with asymptotes $y = \pm 2x$ and directrices $x = \pm 1$;
(iii) the ellipse consisting of all points P such that $|AP| + |BP| = 10$, where A is $(3, 0)$ and B is $(-3, 0)$;
(iv) the parabola with directrix $x + y = 1$ and focus $(-1, -1)$.

Exercise 190 ** Let $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ denote the canonical basis of $\mathbb{R}^2$, and set $\mathbf{e}_1 = (\mathbf{i} + \mathbf{j})/\sqrt{2}$ and $\mathbf{e}_2 = (\mathbf{j} - \mathbf{i})/\sqrt{2}$. Show that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.*

Given $X, Y \in \mathbb{R}$ determine $x, y \in \mathbb{R}$ such that $X\mathbf{e}_1 + Y\mathbf{e}_2 = x\mathbf{i} + y\mathbf{j}$.

Deduce that the curve $x^2 + xy + y^2 = 1$ is an ellipse and find its area.

Exercise 191 *Show that the point $\mathbf{p}(t) = (a\sec t, b\tan t)$ lies on the hyperbola H given by $x^2/a^2 - y^2/b^2 = 1$. Give a domain $S \subset \mathbb{R}$ for t so that $\mathbf{p} : S \to H$ parametrises the hyperbola.*

Write down the equation of the tangent line to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ at the point $(a\sec t, b\tan t)$.

Determine the co-ordinates of the points P and Q where this tangent line meets the asymptotes of the hyperbola, and show that the area of the triangle OPQ is independent of t.

Exercise 192 ** The conic C is formed by intersecting the double cone $x^2 + y^2 = z^2$ with the plane $x + y + z = 1$. Show that the point with position vector*

$$\mathbf{r}(t) = (1 + (\sec t - \tan t)/\sqrt{2},\ 1 + (\sec t + \tan t)/\sqrt{2},\ -1 - \sqrt{2}\sec t)$$

lies on C.

(b) Show that the vectors $\mathbf{e}_1 = (1/\sqrt{6}, 1/\sqrt{6}, -\sqrt{2/3})$ and $\mathbf{e}_2 = (-1/\sqrt{2}, 1/\sqrt{2}, 0)$ are of unit length, are perpendicular to one another and are parallel to the plane $x + y + z = 1$. Show further that

$$\mathbf{r}(t) = (1, 1, -1) + (a\sec t)\mathbf{e}_1 + (b\tan t)\mathbf{e}_2$$

where a and b are positive numbers to be determined.

(c) Show that C has eccentricity $2/\sqrt{3}$, has foci $(1, 1, -1) \pm 2\mathbf{e}_1$ and that the directrices, in parametric form, are $(1, 1, -1) \pm 3\mathbf{e}_1/2 + \lambda\mathbf{e}_2$.

Exercise 193 ** A straight line L through the focus F of a conic C meets C in two points A and B. Show that the quantity*

$$\frac{1}{|AF|} + \frac{1}{|BF|}$$

is independent of the choice of L. [Hint: this question is best done using polar co-ordinates.] What is this quantity for the parabola $y^2 = 4ax$?

Exercise 194 ** Find parametrisations for the following surfaces:*
i) the cylinder $x^2 + y^2 = a^2$, $a > 0$,
ii) the paraboloid $z = x^2 + y^2$,
iii) the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, $a, b, c > 0$,
iv) the hyperboloid of one sheet $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$, $a, b, c > 0$,
v) the surface generated by rotating the curve $y = f(x)$ *about the* x*-axis.*

Exercise 195 ** A torus in* $\mathbb{R}^3$ *has parametrisation*

$$\mathbf{r}(u, v) = ((a + b\cos v)\cos u, (a + b\cos v)\sin u, b\sin v)$$

for $u, v \in (0, 2\pi)$. *Write down an equation for the torus in terms of* x, y, z.

Calculate $\partial\mathbf{r}/\partial u$ *and* $\partial\mathbf{r}/\partial v$. *The torus' surface area is defined by*

$$A = \int_0^{2\pi}\int_0^{2\pi} |\partial\mathbf{r}/\partial u \wedge \partial\mathbf{r}/\partial v| \, \mathrm{d}u \, \mathrm{d}v.$$

Show that the torus has area $4\pi^2 ab$.

Exercise 196 ** Let* $\mathbf{r}(u, v) = \mathbf{s}(a, b)$ *be two different parametrisations for the same surface. Use the chain rule to show that* $\mathbf{r}_u \wedge \mathbf{r}_v$ *and* $\mathbf{s}_a \wedge \mathbf{s}_b$ *are parallel — hence the definition of a normal does not depend on the choice of parametrisation.*

Exercise 197 *Show that there is no function* $f(x, y, z)$ *such that* $\nabla f(x, y, z) = (y, z, x)$.

Exercise 198 *Find the gradient* ∇f *when (i)* $f(x, y, z) = z\sin y - xz$; *(ii)* $f(x, y, z) = ze^x \cos y$.

Exercise 199 *Suppose that a hill has equation*

$$z = 32 - \frac{1}{16}x^2 - \frac{1}{4}y^2$$

where z *is the height above sea-level in metres. Sketch a contour map (i.e. draw on the same axes a set of curves* $z =$ *constant) with various contours shown.*

If you start at the point $(3, 2)$ *and walk in the direction* $\mathbf{i} + \mathbf{j}$ *are you going uphill or downhill? In what direction (in three dimensions!) are you actually travelling? If you are travelling at* v ms^{-1} *what is your vertical speed?*

Exercise 200 *Suppose that the temperature* T *in the* xy*-plane is given by*

$$T(x, y) = xy - x.$$

(a) Sketch, on the same axes, a few isothermal curves (curves of constant temperature).

(b) Find the direction in which the temperature changes most rapidly at $(2, 2)$. *Find the directional derivative of* T *at* $(2, 2)$ *in the direction* $3\mathbf{i} - 4\mathbf{j}$.

(c) Heat flows in the direction $-\nabla T$ *(perpendicular to the isothermals). Express this as a differential equation in* y *and* x *and solve it to say how heat would flow from* $(2, 2)$.

Exercise 201 *Let* f *and* g *be functions of* x, y, z. *Show that*

$$\begin{array}{ll} (i) & \nabla(fg) = f\nabla g + g\nabla f, \\ (ii) & \nabla(f^n) = nf^{n-1}\nabla f, \\ (iii) & \nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}. \end{array}$$

(iv) The Laplacian ∇^2 *is defined by*

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Show that

$$\nabla^2(fg) = f\nabla^2 g + 2(\nabla f)\cdot(\nabla g) + g\nabla^2 f.$$

(v) Let $\mathbf{a}$ *be a constant vector and let* $\mathbf{r} = (x, y, z)$. *Calculate* $\nabla(\mathbf{a}\cdot\mathbf{r})$.

Exercise 202 *Let* $\mathbf{r} = (x, y, z)$ *and* $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$*. Show that for functions* $f : \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R}^3 \to \mathbb{R}$

$$\nabla (f (g (\mathbf{r}))) = f' (g (\mathbf{r})) \nabla g (\mathbf{r}) .$$

Using this result (with $g (\mathbf{r}) = r$*), find* ∇F *when*

$$\text{(i)} \qquad F (\mathbf{r}) = r, \qquad \text{(ii)} \qquad F (\mathbf{r}) = \ln r, \qquad \text{(iii)} \qquad F (\mathbf{r}) = \frac{1}{r}.$$

Exercise 203 *Show that in terms of spherical polar co-ordinates*

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta,$$

where $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$ *are as given in Exercise 162.*

Exercise 204 ** Given orthogonal curvilinear co-ordinates in the plane (see Exercise 164) show that*

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial x_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial x_2}\mathbf{e}_2.$$

Exercise 205 *Let* $f (x, y) = x^2 y^3$*. What is* ∇f*? Determine*

$$\lim_{t \to 0} \frac{f (\mathbf{r} + t\mathbf{n}) - f (\mathbf{r})}{t}$$

where $\mathbf{r} = (1, 1)$ *and* $\mathbf{n} = (a, b)$ *is a unit vector so that* $a^2 + b^2 = 1$*. What is the maximum value of the limit that can be obtained by varying the vector* $\mathbf{n}$*?*

Exercise 206 *Find a unit vector normal to the surface* $xy^2 + 2yz = 4$ *at the point* $(-2, 2, 3)$*.*

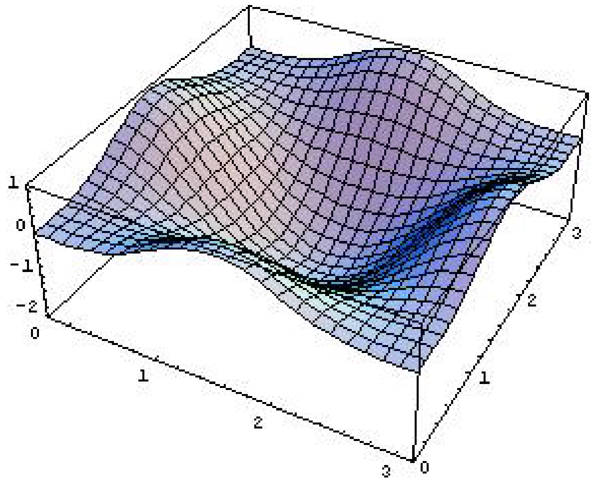
Exercise 207 *Find the angle between the sphere* $x^2 + y^2 + z^2 = 2$ *and the cylinder* $x^2 + y^2 = 1$ *at any point where they intersect.*

11. CRITICAL POINTS

In this final chapter we use the gradient vector to find the *critical points* (or *stationary points*) of a real function. The idea of a critical point is the multivariate generalization of a stationary point of a one-variable function, but critical points in several variables can arise with a more complicated variety. Consider the function

$$k(x,y) = \sin^4 x + \sin^4 y - 4\sin^2 x \sin^2 y \tag{11.1}$$

and its graph $z = k(x,y)$ (see below) for the range $0 \leqslant x, y \leqslant \pi$.



The function has critical points of different kinds as we shall see. The most obvious, and one easily defined, is the minimum in the middle at $(\pi/2, \pi/2)$.

Definition 165 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function. Then f has a* ***local minimum*** *at $\mathbf{p} = (p_1, \ldots, p_n)$ if there exists $\varepsilon > 0$ such that*

$$f(\mathbf{x}) \geqslant f(\mathbf{p}) \quad \textit{whenever} \quad |x_i - p_i| < \varepsilon \textit{ for each } i.$$

That is, if there is a small n-dimensional cube (or equivalently n-dimensional ball) about $\mathbf{p}$ such that f gets no smaller than at $\mathbf{p}$ in this cube.

Similarly we define f to have a ***local maximum*** *at $\mathbf{p} = (p_1, \ldots, p_n)$ if there exists $\varepsilon > 0$ such that*

$$f(\mathbf{x}) \leqslant f(\mathbf{p}) \quad \textit{whenever} \quad |x_i - p_i| < \varepsilon \textit{ for each } i.$$

It may, of course, be the case that f gets greater/smaller at other points in $\mathbb{R}^n$. But we can define:

Definition 166 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function. Then f has a* ***global minimum*** *at $\mathbf{p} = (p_1, \ldots, p_n)$ if*

$$f(\mathbf{x}) \geqslant f(\mathbf{p}) \ \textit{ for all } \mathbf{x} \in \mathbb{R}^n,$$

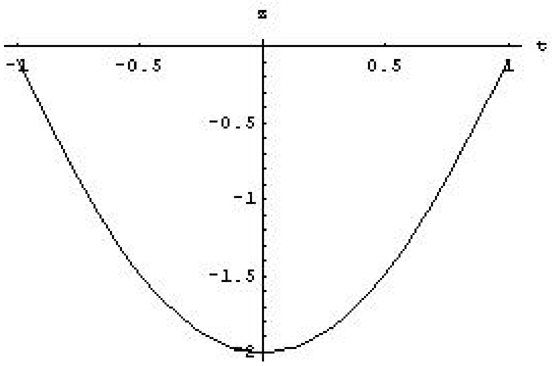
and f has a ***global maximum*** *at $\mathbf{p} = (p_1, \ldots, p_n)$ if*

$$f(\mathbf{x}) \leqslant f(\mathbf{p}) \ \textit{ for all } \mathbf{x} \in \mathbb{R}^n.$$

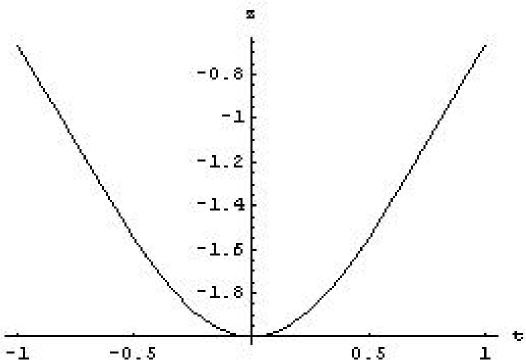
Suppose now that $n = 2$ and that f has partial derivatives of all orders with respect to x and y. If f has a local minimum at $\mathbf{p}$ then in whatever direction $\mathbf{u} = (u_1, u_2)$ we move from $\mathbf{p}$ the function cannot decrease. If we take the cross-section of the graph $z = f(x,y)$ by the vertical plane through $\mathbf{p}$ and parallel to the vector $\mathbf{u}$ we would have a graph in the tz-plane of

$$z = f(p_1 + tu_1, p_2 + tu_2)$$

and we know that at such a minimum $\mathrm{d}z/\mathrm{d}t = 0$. Below are graphs of $z = k\left(p_1 + tu_1, p_2 + tu_2\right)$ for the function (11.1) with $\mathbf{p} = (\pi/2, \pi/2)$ and different values of $\mathbf{u}$.



$\mathbf{u} = (1, 0)$



$\mathbf{u} = \left(1/\sqrt{2}, 1/\sqrt{2}\right)$

Recall from the chain rule that

$$0 = \left.\frac{\mathrm{d}z}{\mathrm{d}t}\right|_{t=0} = u_1 \frac{\partial f}{\partial x}(\mathbf{p}) + u_2 \frac{\partial f}{\partial y}(\mathbf{p}) = \mathbf{u} \cdot \nabla f(\mathbf{p})$$

is the directional derivative of f at $\mathbf{p}$ in the direction $\mathbf{u}$. As this is true for all vectors $\mathbf{u}$ then in particular it is true when $\mathbf{u} = \nabla f(\mathbf{p})$, and hence it follows that $|\nabla f(\mathbf{p})|^2 = 0$ and so $\nabla f(\mathbf{p}) = \mathbf{0}$. This motivates our next definition:

Definition 167 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function which is differentiable with respect to each variable. A point p is said to be a* ***critical point*** *and $f(p)$ is said to be a* ***critical value*** *if*

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \cdots \frac{\partial f}{\partial x_n}\right) = 0.$$

It's clear from the above that local maxima and minima are critical points but there are other types.

Example 168 *Determine the critical points of*

$$k(x, y) = \sin^4 x + \sin^4 y - 4\sin^2 x \sin^2 y.$$

Solution. We have

$$\begin{aligned}
\partial k/\partial x &= 4\sin^3 x \cos x - 8\sin x \cos x \sin^2 y = \sin 2x\left(-1 - \cos 2x + 2\cos 2y\right), \\
\partial k/\partial y &= 4\sin^3 y \cos y - 8\sin^2 x \sin y \cos y. = \sin 2y\left(-1 - \cos 2y + 2\cos 2x\right).
\end{aligned}$$

There are various cases to consider.
(i) $\sin 2x = 0$ and $\sin 2y = 0$; so $(x, y) = (m\pi/2, n\pi/2)$ for integers $m, n \in \mathbb{Z}$.
(ii) $\sin 2x = 0$, $\cos 2x = 1$ and $\cos 2y = -1$.
(iii) $\sin 2y = 0$, $\cos 2y = 1$ and $\cos 2x = -1$.
(iv) and in the final case

$$-\cos 2x + 2\cos 2y = 1, \qquad -\cos 2y + 2\cos 2x = 1,$$

so that $\cos 2x = \cos 2y = 1$. Note that these last three cases are already included in case (i). ■

The function $k(x, y)$ is period with period π independently in the two variables. If we sketch the graph of

$$z = \sin^4 x + \sin^4 y - 4\sin^2 x \sin^2 y \quad \text{for the range} \quad 0 \leqslant x, y < \pi$$

we see that the four critical points in this range

$$(0, 0), (\pi/2, 0), (0, \pi/2), (\pi/2, \pi/2)$$

are quite different in nature. Let's consider them separately:–

- $(0,0)$: At this point k is 0. We can get a sense of how k behaves near $(0,0)$ by moving off in various directions from $(0,0)$. If we move along the x-axis we see

$$k(t,0) = \sin^4 t,$$

a cross-section of the graph which is a minimum. However if we move along the $y = x$ line we note

$$k(t,t) = -2\sin^4 t,$$

a cross-section which yields a maximum. So depending on the direction we move from $(0,0)$ the function may increase or decrease. In this case $(0,0)$ is an example of a *saddle point*.

- $(\pi/2, 0)$: At this point k is 1. If we move along the x-axis we see

$$k(\pi/2 + t, 0) = \cos^4 t,$$

a cross-section of the graph which is a maximum. If we move parallel to the y-axis we also note

$$k(\pi/2, t) = 1 + \sin^4 t - 4\sin^2 t,$$

which is also a maximum as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} k(\pi/2, t) = 12\sin^2 t\cos^2 t - 4\sin^4 t + 8\sin^2 t - 8\cos^2 t = -8 < 0 \quad \text{when } t = 0.$$

In fact we shall see that however we proceed from $(\pi/2, 0)$ the function decreases. By symmetry the same is true at $(0, \pi/2)$

- $(\pi/2, \pi/2)$: At this point k is -2. If we move along the x-axis we see

$$k(\pi/2 + t, \pi/2) = \cos^4 t + 1 - 4\cos^2 t$$

which is a cross-sectional minimum as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} k(\pi/2 + t, \pi/2) = 12\cos^2 t\sin^2 t - 4\cos^4 t + 8\cos^2 t - 8\sin^2 t = 4 > 0 \quad \text{when } t = 0.$$

In fact we shall see that however we proceed from $(\pi/2, \pi/2)$ the function increases.

So suppose we are in the general situation of having a graph $z = g(x,y)$ with a critical point at $\mathbf{p} = (p_1, p_2)$. Consider the function

$$h(t) = g(p_1 + u_1 t, p_2 + u_2 t)$$

where $\mathbf{u} = (u_1, u_2)$ is a unit vector. A vertical planar cross-section of z through $\mathbf{p}$ and parallel to $\mathbf{u}$ would yield a graph of $z = h(t)$. By the second-order chain rule we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} h''(t)\bigg|_{t=0} = (u_1)^2 g_{uu}(\mathbf{p}) + 2u_1 u_2 g_{uv}(\mathbf{p}) + (u_2)^2 g_{vv}(\mathbf{p}). \tag{11.2}$$

We are interested in the sign of (11.2) as we consider all possible unit vectors $\mathbf{u}$. In the case of a maximum (resp. minimum) the above should always be negative (resp. positive) whilst in the case of a saddle point both signs should be possible. If $u_2 \neq 0$ we can rearrange (11.2) as

$$(u_2)^2 \left[\left(\frac{u_1}{u_2}\right)^2 g_{uu}(\mathbf{p}) + 2\left(\frac{u_1}{u_2}\right) g_{uv}(\mathbf{p}) + g_{vv}(\mathbf{p}) \right]$$

then we have $(u_2)^2 > 0$ and the second bracket is a quadratic in u_1/u_2. If we consider the discriminant (i.e. "$b^2 - 4ac$") we see it equals

$$4\left(g_{uv}(\mathbf{p})^2 - g_{uu}(\mathbf{p})\, g_{vv}(\mathbf{p})\right)$$

and so the second bracket:

- is always positive if $g_{uv}(\mathbf{p})^2 < g_{uu}(\mathbf{p})\,g_{vv}(\mathbf{p})$ and $g_{uu}(\mathbf{p}) > 0$ (the second bracket represents a $\cup$-shaped parabola with no real roots);
- is always negative if $g_{uv}(\mathbf{p})^2 < g_{uu}(\mathbf{p})\,g_{vv}(\mathbf{p})$ and $g_{uu}(\mathbf{p}) < 0$ (the second bracket represents a $\cap$-shaped parabola with no real roots);
- takes positive and negative values if $g_{uv}(\mathbf{p})^2 > g_{uu}(\mathbf{p})\,g_{vv}(\mathbf{p})$ (the second bracket represents a parabola which crosses the horizontal axis twice).

Note that if $u_2 = 0$ then we have a further cross-sectional minimum or maximum to add to the first or second case depending on whether $g_{uu}(\mathbf{p})$ is positive or negative respectively.

Definition 169 *Let $\mathbf{p} \in \mathbb{R}^2$ be a critical point of the function $g : \mathbb{R}^2 \to \mathbb{R}$. We say that $\mathbf{p}$ is a **non-degenerate** critical point if*

$$g_{uv}(\mathbf{p})^2 \neq g_{uu}(\mathbf{p})\,g_{vv}(\mathbf{p}).$$

Remark 170 It is beyond the scope of this chapter to discuss degenerate critical points at any length though examples of such are discussed in Examples 173 and 175.

So in the prequel to Definition 169 we showed:

Theorem 171 *If $\mathbf{p} \in \mathbb{R}^2$ is a non-degnerate critical point of the function $g : \mathbb{R}^2 \to \mathbb{R}$ then:*
(i) $\mathbf{p}$ is a maximum if $g_{uv}(\mathbf{p})^2 < g_{uu}(\mathbf{p})\,g_{vv}(\mathbf{p})$ and $g_{uu}(\mathbf{p}) < 0$
(ii) $\mathbf{p}$ is a minimum if $g_{uv}(\mathbf{p})^2 < g_{uu}(\mathbf{p})\,g_{vv}(\mathbf{p})$ and $g_{uu}(\mathbf{p}) > 0$
(iii) $\mathbf{p}$ is a saddle point if $g_{uv}(\mathbf{p})^2 > g_{uu}(\mathbf{p})\,g_{vv}(\mathbf{p})$.

Example 172 *Find the critical points of the function*

$$f(x,y) = x^2 + 2xy - y^2 + y^3,$$

and classify them as maxima, minima or saddle points.

Solution. Firstly the simultaneous equations

$$f_x = 2x + 2y = 0, \qquad f_y = 2x - 2y + 3y^2 = 0,$$

imply $3y^2 = 4y$ and hence the function has two critical points $(0,0)$ and $(-4/3, 4/3)$. Now

$$f_{xx} = 2, \qquad f_{xy} = 2, \qquad f_{yy} = 6y - 2.$$

So, to classify the critical points we note:

Point	f_{xx}	f_{xy}	f_{yy}	$f_{xx}f_{yy} - f_{xy}^2$	Type
$(0,0)$	2	2	-2	$-8 < 0$	saddle,
$(-4/3, 4/3)$	2	2	10	$16 > 0$	minimum.

■

Example 173 *Returning to the function*

$$k(x,y) = \sin^4 x + \sin^4 y - 4\sin^2 x \sin^2 y$$

from (11.1) we were investigating its critical points at $(0,0)$, $(\pi/2, 0)$ and $(\pi/2, \pi/2)$. Note that

$$\begin{aligned}
k_x &= 4\sin^3 x\cos x - 8\sin x\cos x\sin^2 y,\\
k_y &= 4\sin^3 y\cos y - 8\sin y\cos y\sin^2 x,\\
k_{xx} &= 12\sin^2 x\cos^2 x - 4\sin^4 x - 8\cos^2 x\sin^2 y + 8\sin^2 x\sin^2 y,\\
k_{xy} &= -16\sin y\cos y\sin x\cos x,\\
k_{yy} &= 12\sin^2 y\cos^2 y - 4\sin^4 y - 8\cos^2 y\sin^2 x + 8\sin^2 y\sin^2 x.
\end{aligned}$$

So, classifying the above three critical points we see:

Point	k_{xx}	k_{xy}	k_{yy}	$k_{xx}k_{yy} - k_{xy}^2$	*Type*
$(0,0)$	0	0	0	0	*degenerate*
$(\pi/2, 0)$	-4	0	-8	$32 > 0$	*maximum,*
$(\pi/2, \pi/2)$	4	0	4	$16 > 0$	*minimum.*

Remark 174 Even though $(0,0)$ is degenerate, we can easily eliminate it as a possible maximum or minimum by our earlier analysis after Example 168 where we showed that the function both increased and decreased in different directions from $(0,0)$.

Example 175 **Find an classify the critical points of*

$$f(x,y) = x^2 - y^4, \quad g(x,y) = x^2 y + xy^2, \quad h(x,y) = x^4 + 2x^2y^2 + 2y^4.$$

Solution. (i) We have $f_x = 2x = 0$ and $f_y = -4y^3 = 0$ only at $(0,0)$. At $(0,0)$ we have $f_{xx} = 2,\ f_{xy} = 0,\ f_{yy} = 0$ and so $(0,0)$ is degenerate. However we can easily see that moving along the x-axis we perceive $(0,0)$ as a minimum as $f(x,0) = x^2$, whilst if we move along the y-axis we perceive $(0,0)$ as a maximum as $f(0,y) = -y^4$. So $(0,0)$ is a saddle point.

(ii) Note $g_x = 2xy + y^2 = 0$ and $g_y = x^2 + 2xy = 0$ only at $(0,0)$. At $(0,0)$ we have $g_{xx} = g_{xy} = g_{yy} = 0$ and so $(0,0)$ is degenerate. But it is very easy to see that $g(x,y)$ takes both positive and negative values near 0. As $g(x,y) = xy(x+y)$ then $g(x,y) > 0$ in the region $x > 0, y > 0, x + y > 0$ (say at points $(\varepsilon, \varepsilon)$ where $\varepsilon > 0$, which can be arbitrarily close to the origin), but $g(x,y) < 0$ in the region $x > 0, y < 0, x + y > 0$ (say at points $(2\varepsilon, -\varepsilon)$ where $\varepsilon > 0$).

(iii) Note $h_x = 4x^3 + 4xy^2 = 0$ and $h_y = 4x^2y + 8y^3 = 0$ only at $(0,0)$. At $(0,0)$ we have $h_{xx} = h_{xy} = h_{yy} = 0$ and so $(0,0)$ is degenerate. We can, however, easily see that

$$h(x,y) = \left(x^2 + y^2\right)^2 + y^4 \geqslant 0 \text{ for all } (x,y) \in \mathbb{R}^2$$

and so $(0,0)$ is a minimum, is in fact a global minimum of $h(x,y)$. ■

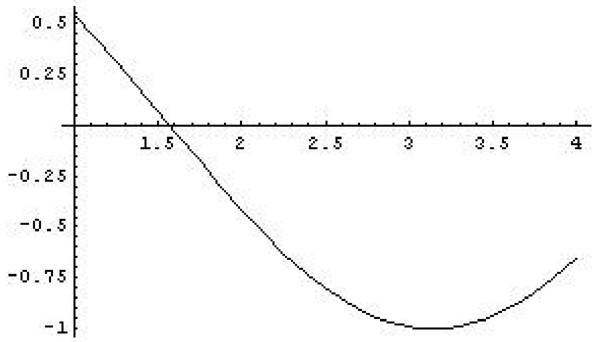
Recall that when asked to find the extreme values of a function on a region of the real line (e.g. an interval) the maxima and minima may arise in one of two ways:

- at **internal** extrema — this is when the function takes its maximum or minimum at a point in (a,b) and in this case $f'(x) = 0$ at such a point;
- at **external** extrema — this is when the function takes its maximum or minimum at either $x = a$ or $x = b$ and it need no longer be the case that $f'(x) = 0$.

By way of example consider the function

$$f(x) = \cos x \quad \text{for } 1 \leqslant x \leqslant 4.$$

It is clear, from the graph below, that the function takes its maximum (in this interval) at $x = \pi/2$ which is an internal maximum, and takes its minimum at $x = 4$ which is an external maximum.



$y = \cos x$ for $1 \leqslant x \leqslant 4$.

There are various ways to approach the problem below, including purely geometric ones, but we will treat this as an extremum problem of a two variable problem.

Example 176 *Given a triangle with angles A, B, C, none of which is obtuse, show that*

$$2 < \sin A + \sin B + \sin C \leqslant \frac{3\sqrt{3}}{2}.$$

Solution. At first glance this may seem like a three variable problem but of course $A + B + C = \pi$ and so in fact all possibilities are determined by A and B alone. Now (A, B) cannot take all possible values in the plane; rather it must be the case that

$$0 < A < \frac{\pi}{2}, \qquad 0 < B < \frac{\pi}{2},$$

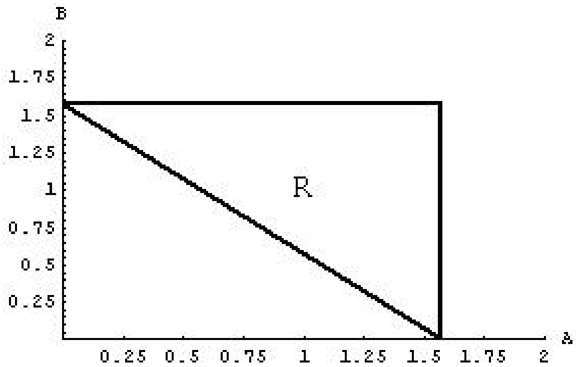
and as $C = \pi - A - B$ is also acute it follows that

$$\frac{\pi}{2} < A + B < \pi.$$

These three inequalities mean that we only need to treat those triangles (A, B) which lie in the interior of

$$R = \left\{(A, B) : 0 \leqslant A, B \leqslant \frac{\pi}{2} \leqslant A + B \leqslant \pi\right\}$$

which is a triangular region of the AB-plane sketched in the diagram below.



So our problem then is to find the maximum and minimum value taken

$$f(A, B) = \sin A + \sin B + \sin(\pi - A - B) = \sin A + \sin B + \sin(A + B)$$

in the region R.

- **Internal Extrema:** At the internal extrema $f_A = f_B = 0$ and so we have

$$\frac{\partial f}{\partial A} = \cos A + \cos(A + B) = 0, \qquad \frac{\partial f}{\partial B} = \cos B + \cos(A + B) = 0.$$

  These two equations hold when

$$\cos A = \cos B = -\cos(A + B) = \cos(\pi - A - B)$$

  which means that $A = B = \pi - A - B$ for the allowed values of A, B. In short, it means that the triangle is equilateral. For such triangles $f = 3\sqrt{3}/2$, which is the maximum value that f can take inside R. This is the only internal extremum.

- **External Extrema:** instead of having just two endpoints to check, as in the one-variable case, our boundary now comprises three sides of a triangle. These are given by:

$$\begin{aligned} A &= \pi/2, \qquad 0 \leqslant B \leqslant \pi/2, \\ B &= \pi/2, \qquad 0 \leqslant A \leqslant \pi/2, \\ A + B &= \pi/2, \qquad 0 \leqslant A, B \leqslant \pi/2. \end{aligned}$$

We will treat the first side and set $A = \pi/2$. Then

$$\begin{aligned} f\left(\frac{\pi}{2}, B\right) &= \sin\frac{\pi}{2} + \sin B + \sin\left(\frac{\pi}{2} + B\right) \\ &= 1 + \sin B + \cos B \\ &= 1 + \sqrt{2}\cos\left(B - \frac{\pi}{4}\right). \end{aligned}$$

We can see then that, as B varies over $[0, \pi/2]$ then f is minimal at $B = 0$ and $\pi/2$ at which points $f = 2$ and has a maximum of $1 + \sqrt{2}$ on this side (and this is less than $3\sqrt{3}/2$. Note that when $A = \pi/2$ and $B = 0$ or $\pi/2$ we don't really have a triangle, in the normal sense of the word — rather we have two parallel lines which meet at infinity.

If we now wished we could deal with the remaining two sides similarly, but it is easier to note that they will be the same by the symmetry of the situation — having a right angle at A will result in triangles congruent to those arising from having a right angle at B or C.

So finally we have our maximum and minimum for f, and noting that the minimum is not actually achieved by a proper triangle we have

$$2 < \sin A + \sin B + \sin C \leqslant \frac{3\sqrt{3}}{2}$$

for any triangle with no obtuse angles.

■

11.1 Exercises

Exercise 208 *Find and classify the critical points of* $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$.

Exercise 209 *Find and classify the critical points of* $f(x, y) = x^3 + y^3 + 3xy$.

Exercise 210 *Find and classify the critical points of* $f(x, y) = xy\left(x^2 + y^2 - 1\right)$.

Exercise 211 *Find and classify the critical points of* $f(x, y) = \sin^2 x + \sin^2 y - \cos^2 x \cos^2 y$.

Exercise 212 *Find the stationary values of*

$$f(x, y) = x^3 + x^2 + 2\alpha xy + y^2 + 2\alpha x + 2y$$

(a) when $\alpha > 1$, *(b) when* $\alpha < 1$,
and classify them as maxima, minima or saddle points.

Exercise 213 *Distinguishing cases determine the number of critical points of*

$$f(x, y) = x^3 - 12xy + 48x + cy^2,$$

where $c \in \mathbb{R}$.

Exercise 214 *Find and classify the critical points of* $f(x, y) = 5x + 15y - \left(x^2 + y^2\right)^2$.

Exercise 215 *By making a change to polar co-ordinates, find the maximum and minimum values of* $x^2 + y^2$ *on the curve with equation*

$$x^2 + 2xy + 2y^2 = 8.$$

Exercise 216 *By considering* ∇f *where* $f(x, y) = x^2/4 + y^2$, *or otherwise, find the shortest distance from the line* $x + y = 3$ *to the ellipse* $x^2/4 + y^2 = 1$.

Exercise 217 *By using appropriate parameters find the maximum of* $f(x,y,z) = x^3 + y^3 + z^3$
(i) subject to $x^2 + y^2 + z^2 = 1$*;*
(ii) subject to $x^2 + y^2 + z^2 = 1$ *and* $x + y + z = 0$.

Exercise 218 *Find the greatest value of* $f(x,y) = 1 + xy + yz$ *on the unit sphere* $x^2 + y^2 + z^2 = 1$.

Exercise 219 ** Show that if* $x, y, z > 0$ *then*

$$(xyz)^{1/3} \leqslant \frac{x+y+z}{3}.$$

Deduce for $a, b, c > 0$ *that*

$$ab^2c^3 \leqslant 108\left(\frac{a+b+c}{6}\right)^6.$$

Exercise 220 ** (a) Let* $a, b, c \in \mathbb{R}$. *Show that* $f(x,y) = ax^2 + 2bxy + cy^2$ *has a critical point at the origin. Show further that*

$$f(x,y) = (x,y)\,A\binom{x}{y}$$

where

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

(b) Show that the matrix A *has two real eigenvalues* λ_1, λ_2 *and that there are orthonormal vectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *such that*

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i \quad \text{for } i = 1, 2.$$

(c) Show that the origin is a minimum, maximum or saddle respectively if λ_1, λ_2 *are both positive, both negative or of mixed signs.*

Exercise 221 ** The point* $\mathbf{x}_0 = (x_0, y_0, z_0)$ *is a maximum or minimum of a smooth function* $f(\mathbf{x})$, *where* $\mathbf{x} = (x, y, z)$, *subject to the constraint that* $\mathbf{x}$ *lies on the surface* $g(\mathbf{x}) = 0$. *Let* $\mathbf{r}(t) = (x(t), y(t), z(t))$ *be a smooth curve which lies in the surface* $g(\mathbf{x}) = 0$ *and such that* $\mathbf{r}(0) = \mathbf{x}_0$.

Write $F(t) = f(\mathbf{r}(t))$. *Explain why* $F'(0) = 0$ *for all such curves and deduce that* $\nabla f(\mathbf{x}_0)$ *is normal to* $g(\mathbf{x}) = 0$ *at* $\mathbf{x}_0$. *Hence show that there is a constant* λ *such that*

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0).$$

This is known as the ***method of Lagrangian multipliers.***

Exercise 222 ** Use Lagrangian multipliers to find the shortest distance from the origin to the surface*

$$x = yz + 10.$$

Exercise 223 ** Use Lagrangian multipliers to find the shortest distance from the origin to the line of intersection of the planes* $2x + y - z = 1$ *and* $x - y + z = 2$. *Repeat the calculation using geometric methods.*

Exercise 224 ** Use Lagrangian multipliers to find the largest triangle that can be inscribed in the ellipse* $x^2/a^2 + y^2/b^2 = 1$.